ENTWURF EINER VERALLGEMEINERTEN RELATIVITÄTSTHEORIE UND EINER THEORIE DER GRAVITATION

I. PHYSIKALISCHER TEIL

VON

ALBERT EINSTEIN

IN ZÜRICH

II. MATHEMATISCHER TEIL

VON

MARCEL GROSSMANN

IN ZÜRICH

LEIPZIG UND BERLIN

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Outline of a Generalized Theory of Relativity and of a Theory of Gravitation

or

"*The Description of a Miracle*",
God would say

by
Albert Einstein

Marcel Grossmann

Part I. Physical Part, page 4, by Albert Einstein

Part II. Mathematical Part, page 31, by Marcel Grossmann

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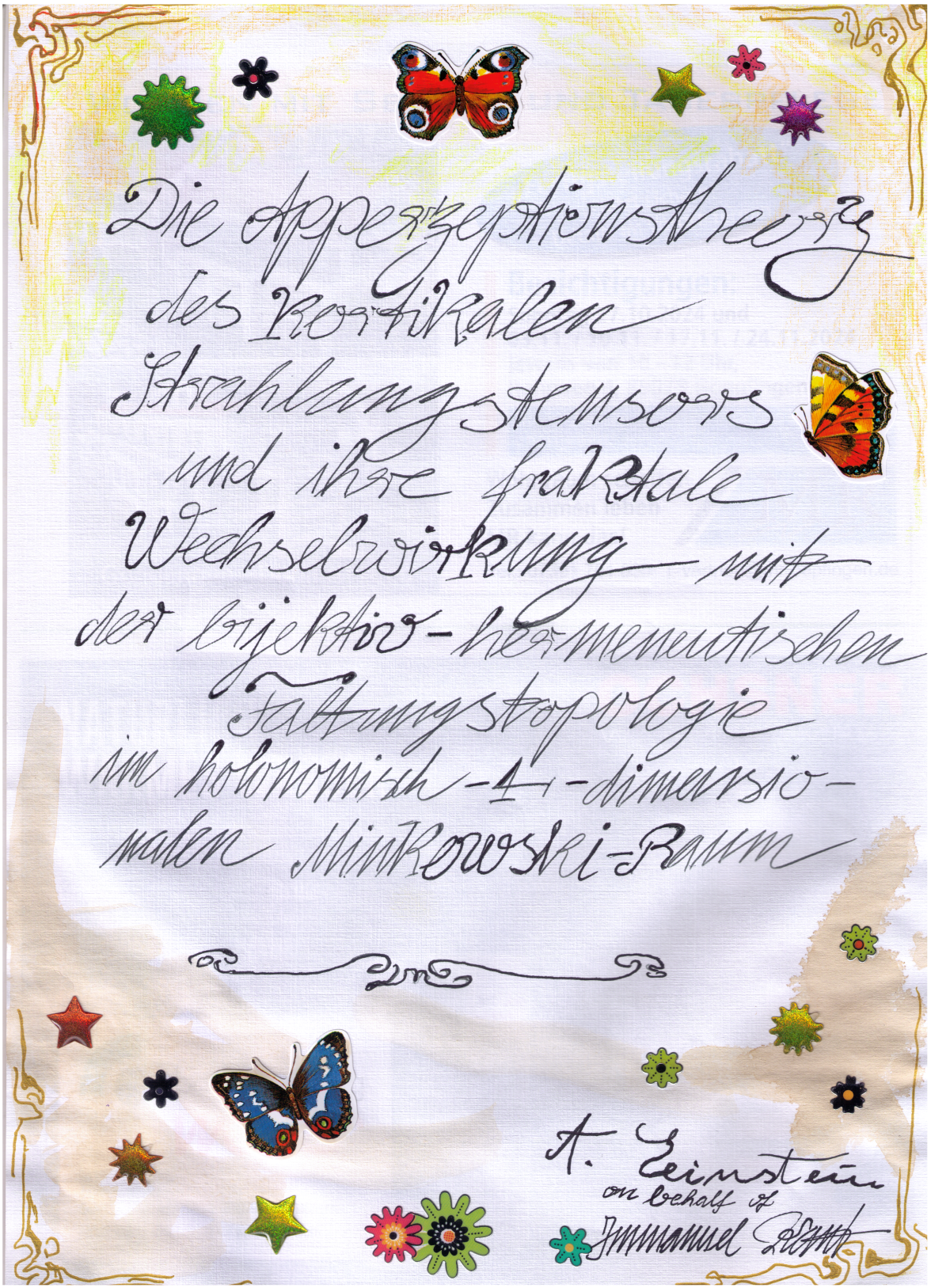
Cortex
Immanuel
Kant's
Cat


Die Apperzeptionstheorie
des kortikalen
Strahlungssensors
und ihre fraktale
Wechselwirkung mit
der objektiv-hermeneutischen
Fallungstopologie
im holonomisch-1-dimensio-
nalen Minkowski-Raum
A. Einstein
on behalf of
Immanuel Kant

Life is an ever going search for the quest of transcendence of Being and Time, in a holonomic Heideggerian sense. Kant may say "True knowledge or enlightenment is to be found deeply buried in the topological fissures of a Reebian brain foliation in 4-dimensional fractal transcendental philosophy." and Einstein may respond that HE plays dice, but could also stochastically scramble fractalizationly with a cube-shaped metric the curved pseudo-Riemannian manifold we are sailing on. Anyway. . . you are not allowed to know everything. Life and ultimate knowledge are two excluding principles. If you know everything you do not exist anymore. Not in this entity called universe. The somehow frustrating thing about that is that if you're in a state of knowing everything, you cannot share it with any human being alive. But sharing things on the other hand is something beautiful and makes you feeling happy. So, the question may arise: "Is divine knowledge making you happy?" Hermann Hesse says: "The answer lies within [the darkness of] yourself."

Jan

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Once there was sucking darkness, and God told me “Son, do as I have planned for you”, and without knowing how. . . I did

Abstract

A critique of the treatise is beyond the scope of these presentations. Suffice it to say that it concerns the theory of gravity developed by A. EINSTEIN in recent years, [4, 5], and presented at the Vienna Natural Scientists' Conference in the fall of 1913. It is based on the "equivalence hypothesis", according to which all laws of nature are the same for a uniformly accelerated coordinate system and for an acceleration-free one located in a corresponding homogeneous gravitational field. The experimental basis for this hypothesis is the constancy of the ratio of inertial and gravitational mass, which has been validated with great accuracy by the experiments of LORÁND EÖTVÖS. (The direction of the plumb line, as the resultant of the actual gravity and the centrifugal force, would be different for bodies of different mass ratios; this would require a torque to be exerted on a sensitive torsion balance to which the two bodies are attached, the absence of which was confirmed in EÖTVÖS' experiments.)
The theory requires new mathematical tools; these are developed in the second part of the treatise by M. GROSSMANN.

With R. WULKENHAAR, [16] we summarize:
. . . EINSTEIN was aware of the shortcomings of the special theory of relativity. Physics should be identical not only in all inertial reference frames (which cannot be meaningfully defined within the framework of the special theory of relativity!), but also in arbitrarily moving reference frames. This *general principle of relativity* gave the theory its name. Over time, however, it became clear that this was not the core of the theory. The principle simply states that physics must be geometrized and described *by intrinsic objects on manifolds*. Manifolds are defined by an atlas of local maps, together with a conversion rule in the overlap area. Each map is a representation of actual physics in a chosen reference system. Experiments only ever provide this dedicated map description, and an abstraction process must be carried out in order to infer the intrinsic reality on a manifold. In practice, this can be difficult. From a theoretical point of view, however, it is an enormous simplification: *formulate the laws of nature on manifolds*. Then:

- The general principle of relativity is an empty statement.
- One has the freedom to choose a particularly meaningful map in which the calculation is simple. One gains predictions for experiments.
- Geometrization is a strong restriction on possible laws of nature; one therefore has to try fewer possibilities.

Through profound considerations, EINSTEIN came to the conclusion in 1907 that this geometrization, together with the *equivalence principle*, i.e., the universality of free fall, says something about gravity. This equivalence principle is the actual core of the entire theory; everything else is mathematical craftsmanship. We will limit ourselves to the conclusion:

There is no such thing as gravitational force!

There is free fall, i.e., force-free geodesic motion in pseudo-RIEMANNian manifolds. The moon's orbit, like NEWTON's apple, is nothing more than free fall towards the Earth, only with different initial data. This free fall is the natural motion, and no force is required for it! Force is only needed if one wants to slow down or completely prevent free fall. The force commonly referred to as gravitational force is the same illusion as centrifugal force. Both forces must be applied to divert the body in question from its natural motion, i.e., to force an acceleration relative to geodesic motion.
. . . After these preliminary remarks, we can conclude (without proof):

Theorem (Equivalence principle)**.** *Local experiments of a certain type with test bodies are not able to distinguish whether the experiment is being conducted in a laboratory that is falling freely in a gravitational field or whether there exists no gravitational field in the laboratory.*
The laws of physics can therefore be transformed locally into a reference system in which the special theory of relativity applies.

An equivalent statement would be:

Theorem (Equivalence principle)**.** *The equivalence principle forms the basis of general relativity. It states that a homogeneous gravitational field is equivalent to a uniformly accelerated reference frame, i.e. that an observer in a closed room with no information from outside cannot determine by any experiment whether he is in zero gravity or in free fall. The equations of motion are therefore the same in both systems.*

The *principle of covariance* follows from the equivalence principle.
It states that, based on the special theory of relativity, the quantities and relationships or laws such as vectors of the four-stream density, the LEVI-CIVITA symbol, and contravariant differentials are tensors in any coordinate system and, in the special case of the MINKOWSKI system, revert to the familiar Lorentz tensors of special relativity without the influence of gravity. The validity of the laws in any coordinate system is ensured because all coordinate systems are equal as long as the laws are formulated as tensor laws. This also applies when the curved coordinates describe a curved space, i.e., under the influence of gravity.

Ultimately, the fundamental statement holds true that gravity, which is caused by a mass embedded in space, can be attributed to a geometric phenomenon of a curved 4-dimensional pseudo-RIEMANNian manifold (or space-time manifold, respectively).

Part I.

Physical Part

by ALBERT EINSTEIN

1. Introduction – General Remarks

The theory expounded in what follows derives from the conviction that the proportionality between the inertial and the gravitational mass of bodies is an exactly valid law of nature that must already find expression in the very foundation of theoretical physics. I already sought to give expression to this conviction in several earlier papers by seeking to reduce the gravitational mass to the inertial mass; this endeavor led me to the hypothesis that, from a physical point of view, an (infinitesimally small extended, homogeneous) gravitational field can be completely replaced physically by a state of acceleration of the reference system. This hypothesis can be expressed pictorially in the following way: An observer enclosed in a box can in no way decide whether the box is at rest in a static gravitational field, or whether it is in accelerated motion, maintained by forces acting on the box, in a space that is free of gravitational fields (equivalence hypothesis).

We know the fact that the law of proportionality of inertial and gravitational mass is satisfied to an extraordinary degree of accuracy from the fundamentally important investigation by Eötvös, [6, 7], which is based on the following argument. A body at rest on the surface of the Earth is acted upon by gravity as well as by the centrifugal force resulting from Earth's rotation. The first of these forces is proportional to the gravitational mass, the latter to the inertial mass. Thus, the direction of the resultant of these two forces, i.e., the direction of the apparent gravitational force (direction of the plumb) would have to depend on the physical nature of the body under consideration if the proportionality of the inertial and gravitational mass would not be satisfied. In that case the apparent gravitational forces acting on parts of a heterogeneous rigid system can, in general, not be united into a single resultant; instead, in general, there would still remain a torque associated with the apparent gravitational forces that would have to be noticeable if the system would be suspended from a torsion-free thread. By validating the absence of such torques with great accuracy, Eötvös proved that, for the bodies that he investigated, the relationship of the two masses was independent of the nature of the body to such a degree of exactness that the relative difference in this relationship that might still exist from one substance to another must be smaller than one twenty-millionth.

The decomposition of radioactive substances occurs with a release of such significant quantities of energy that the change in the inertial mass of the system which corresponds to that energy decrease according to the theory of relativity is not very small compared to the total mass[1]. In the case of the decay of radium, for example, this decrease amounts to one ten-thousandth of the total mass. If these changes of the inertial mass would not correspond to changes in the gravitational mass, then there would exist deviations of the inertial mass from the gravitational mass to such an extend, far more than those allowed by Eötvös's experiments. Hence it must be considered with a hight probability that the identity of the inertial and gravitational mass is exactly satisfied. For these reasons it seems to me that the equivalence hypothesis, which asserts the essential physical identity of the gravitational with the inertial mass, has a high degree of truth, see also chapter chapter 8.

[1]The decrease of the inertial mass, which corresponds to the released energy E is, as is well known, $\frac{E}{c^2}$, if c is the speed of light.

2. Equations of Motion of the Material Point in a Static Gravitational Field

According to the ordinary theory of relativity, [11], a point moving without the application of any forces moves according to the equation

$$\delta\left\{\int \mathrm{d}s\right\} = \delta\left\{\int \sqrt{-\mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 + c^2 \cdot \mathrm{d}t^2}\right\} = 0\,. \tag{2.1}$$

This equation says nothing other than that the material point moves in a straight line and uniformly. This is the equation of motion in the form of Hamilton's principle; for we can also set

$$\delta\left\{\int \mathcal{H}\,\mathrm{d}t\right\} = 0\,, \tag{2.2}$$

where

$$\mathcal{H} = -\frac{\mathrm{d}s}{\mathrm{d}t} \cdot m$$

and if m corresponds to the rest mass of the material point. From this we obtain, in the familiar way, the momentum J_x, J_y, J_z and the energy E of the moving point:

$$\left.\begin{aligned} J_x &= m \cdot \frac{\partial \mathcal{H}}{\partial \dot{x}} = m \cdot \frac{\dot{x}}{\sqrt{c^2 - \mathbf{q}^2}}\,;\ \text{etc.} \\ \mathrm{E} &= \frac{\partial \mathcal{H}}{\partial \dot{x}} \cdot \dot{x} + \frac{\partial \mathcal{H}}{\partial \dot{y}} \cdot \dot{y} + \frac{\partial \mathcal{H}}{\partial \dot{z}} \cdot \dot{z} - \mathcal{H} = m \cdot \frac{c^2}{\sqrt{c^2 - \mathbf{q}^2}}\,. \end{aligned}\right\} \tag{2.3}$$

This mode of representation differs from the ordinary one only by the fact that in the latter J_x, J_y, J_z and E contain also a factor c. But since c is constant in the ordinary theory of relativity, the system given here is equivalent to the ordinary one. The only difference is that J and E possess dimensions other than those in the customary mode of representation.

I have shown in previous papers that the equivalence hypothesis leads to the consequence that in a static gravitational field the velocity of light c depends on the gravitational potential. This led me to the view that the ordinary theory of relativity provides only an approximation to reality; it should apply only in the limit case where differences in the gravitational potential in the space-time region under consideration are not too great. In addition, I found again Equation 2.1 or Equation 2.2 as the equations of motion of a mass point in a static gravitational field; however, c is not to be conceived in this case as a constant but rather as a function of the spatial coordinates that represents a measure for the gravitational potential. From Equation 2.2 we obtain in the

familiar fashion the equations of motion in the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{m\cdot\dot{x}}{\sqrt{c^2-\mathbf{q}^2}}\right\}=-\frac{mc\cdot\frac{\partial c}{\partial x}}{\sqrt{c^2-\mathbf{q}^2}}\,. \tag{2.4}$$

It is easy to see that the momentum is represented by the same expression as above. In general, Equation 2.3 hold for the material point moving in a static gravitational field. The right-hand side of Equation 2.4 represents the force $\mathcal{F}_x$, exerted on the mass point by the gravitational field. For the special case of rest ($\mathbf{q}=0$) we have[1]

$$\mathcal{F}_x=-m\cdot\frac{\partial c}{\partial x}$$

From this one sees that c plays the role of the gravitational potential.
From Equation 2.3 it follows that for a slowly moving point

$$\left.\begin{aligned} J_x&=\frac{m\dot{x}}{c}\,,\\ \frac{\mathrm{E}}{c}-mc&=\frac{\frac{1}{2}m\mathbf{q}^2}{c}\,. \end{aligned}\right\} \tag{2.5}$$

At a given velocity, the momentum and the kinetic energy are thus inversely proportional to the quantity c; in other words: the inertial mass, as it enters into the momentum and energy, is $\frac{m}{c}$, where m denotes a constant that is characteristic of the mass point and independent of the gravitational potential. This is constant with E. Mach's daring idea that inertia has its origin in an interaction between the mass point under consideration and all of the other mass points; for if we accumulate masses in the vicinity of the mass point under consideration, we thereby decrease the gravitational potential c, thus increasing the quantity $\frac{m}{c}$ that is determinative of inertia.

[1] Remember that Newton's second law is stated as

$$\mathcal{F}=m\cdot a=m\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

with acceleration a, velocity v and m being the mass of the object or material point, respectively.

3. Equations of Motion of the Material Point in an Arbitrary Gravitational Field. Characterization of the Latter

By introducing a spatial variability of the quantity c, we have breached the frame of the theory presently designated as the "relativity theory"; for now the expression designated by ds no longer behaves as an invariant with respect to orthogonal linear transformations of the coordinates. Thus, if the relativity principle is to be maintained –which is beyond doubt– then we must generalize the theory of relativity in such a way that the theory of the static gravitational field whose elements have been indicated above will be contained in it as a special case.

If we introduce a new space-time system K' (x', y', z', t') by means of an arbitrary substitution

$$\begin{aligned} x' &= x'(x, y, z, t) \\ y' &= y'(x, y, z, t) \\ z' &= z'(x, y, z, t) \\ t' &= t'(x, y, z, t)\,, \end{aligned}$$

and if the gravitational field in the original system K was static, then, upon this substitution, Equation 2.1 transforms into an equation of the form

$$\delta \left\{ \int \mathrm{d}s' \right\} = 0\,,$$

where the quantity $\mathrm{d}s'^2$ is set according to

$$= g_{11} \cdot \mathrm{d}x'^2 + g_{22} \cdot \mathrm{d}y'^2 + \cdots + 2g_{12} \cdot \mathrm{d}x'\,\mathrm{d}y' + \cdots$$

and the quantities $g_{\mu\nu}$ are functions of x', y', z', t'. If we put x_1, x_2, x_3, x_4 in place of x', y', z', t', and write again ds instead of ds', then the equations of motion of the material point with respect to K' take the form

$$\left.\begin{aligned} \delta \left\{ \int \mathrm{d}s \right\} &= 0\,, \text{ where} \\ \mathrm{d}s^2 &= \sum_{\mu\nu} g_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu\,. \end{aligned}\right\} \qquad (1'')$$

We thus arrive at the conclusion that, *in general, the gravitational field is characterized by ten*

space-time functions

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \qquad (g_{\mu\nu} = g_{\nu\mu}) \ ,$$

which in the case of the ordinary theory of relativity reduce to

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +c^2 \end{pmatrix},$$

where c denotes a constant. The same kind of decomposition occurs in the static gravitational field of the kind considered above, except that $g_{44} = c^2$ in the latter case is a function of x_1, x_2, x_3.

The Hamiltonian function $\mathcal{H}$ thus has the following value according to[1]

$$\begin{aligned} \mathcal{H} &= -m \cdot \frac{\mathrm{d}s}{\mathrm{d}t} \\ &= -m \cdot \sqrt{g_{11}\dot{x}_1^2 + \cdots + \cdots + 2g_{12}\dot{x}_1\dot{x}_2 + \cdots + \cdots + 2g_{14}\dot{x}_1 + \cdots + \cdots + g_{44}\dot{x}_4^2} \end{aligned} \tag{3.1}$$

in the general case.
The corresponding Lagrangeian equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{H}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{H}}{\partial x} = 0 \tag{3.2}$$

directly result in the expression for the momentum J of the material point and for the force $\mathcal{F}$ caused by the gravitational field exerted on it:

$$\left.\begin{aligned} J_x &= -m \cdot \frac{g_{11}\dot{x}_1 + g_{12}\dot{x}_2 + g_{13}\dot{x}_3 + g_{14}\dot{x}_4}{\frac{\mathrm{d}s}{\mathrm{d}t}} \\ &= -m \cdot \frac{g_{11}\,\mathrm{d}x_1 + g_{12}\,\mathrm{d}x_2 + g_{13}\,\mathrm{d}x_3 + g_{14}\,\mathrm{d}x_4}{\mathrm{d}s} \end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned} \mathcal{F}_x &= -\frac{1}{2}m \cdot \frac{\sum_{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_1}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu}{\mathrm{d}s \cdot \mathrm{d}t} \\ &= -\frac{1}{2}m \cdot \sum_{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_1} \cdot \frac{\mathrm{d}x_\mu}{\mathrm{d}s} \cdot \frac{\mathrm{d}x_\nu}{\mathrm{d}t} \ . \end{aligned}\right\} \tag{3.4}$$

[1] see Part III

Further, for the energy E of the material point the following equation holds:

$$\left.\begin{aligned} -\mathrm{E} &= -\left(\dot{x}\cdot\frac{\partial\mathcal{H}}{\partial\dot{x}}+\cdots+\cdots\right)+\mathcal{H} \\ &= -m\cdot\left(g_{41}\cdot\frac{\mathrm{d}x_1}{\mathrm{d}s}+g_{42}\cdot\frac{\mathrm{d}x_2}{\mathrm{d}s}+g_{43}\cdot\frac{\mathrm{d}x_3}{\mathrm{d}s}+g_{44}\cdot\frac{\mathrm{d}x_4}{\mathrm{d}s}\right). \end{aligned}\right\} \tag{3.5}$$

In the case of the ordinary theory of relativity only linear orthogonal substitutions are permissible. It will turn out that we are able to set up equations for the influence of the gravitational field on the material processes which are covariant with respect to arbitrary substitutions.

First, from the role that ds plays in the law of motion of the material point, we can draw the conclusion that ds must be an absolute invariant (a scalar, i.e. a quantity without direction but with a definite value); from this it follows that the quantities $g_{\mu\nu}$ form a covariant tensor of the second rank[2], which we call the *covariant fundamental tensor*. This tensor determines the gravitational field. Further, it follows from Equation 3.3 and Equation 3.5 that the momentum and the energy of the material point form together a covariant tensor of the first rank, i.e., a covariant vector[3].

[2] cf. chapter 2
[3] cf. chapter 2

4. The Significance of the Fundamental Tensor of the $g_{\mu\nu}$ for the Measurement of Space and Time

From the foregoing, one can already infer that there cannot exist such simple relationships between the space-time coordinates x_1, x_2, x_3, x_4, and the results of measurements obtainable by means of measuring rods and clocks as in the old theory of relativity. With regard to time, this has already found to be true in the case of the static gravitational field, [3]. The question therefore arises, what is the physical meaning (i.e. in the sense of the principle measurability) of the coordinates x_1, x_2, x_3, x_4.

We note that ds is to be understood as an invariant measure of the distance between two infinitely close space-time points. Therefore, ds must also have a physical meaning that is independent of the chosen reference system. We assume that ds is the "naturally measured" distance between the two space-time points and understand this to mean the following:

The immediate vicinity of the point (x_1, x_2, x_3, x_4) with respect to the coordinate system is determined by the infinitesimal variables $\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4$. Now, instead of these infinitesimal variables, we introduce variables in such a way that a linear transformation creates new variables $\mathrm{d}\xi_1, \mathrm{d}\xi_2, \mathrm{d}\xi_3, \mathrm{d}\xi_4$ so that

$$\mathrm{d}s^2 = \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2 - \mathrm{d}\xi_4^2 \,.$$

Within this transformation the $g_{\mu\nu}$ are to be viewed as constants; the real (light) cone $\mathrm{d}s^2 = 0$ appears as related to its principal axes. Then, the ordinary theory of relativity holds true in this elementary $\mathrm{d}\xi_i$-system ($i = 1, 2, 3, 4$), and the physical meaning of lengths and times shall be the same in this system as in the ordinary theory of relativity. Therefore $\mathrm{d}s^2$ is the square of the four-dimensional distance between two infinitely close points in space-time, measured by a rigid body that is not accelerated in the $\mathrm{d}\xi_i$ system and by unit scales and clocks arranged at rest relative to this body.

From this, one sees that for given $\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4$, the natural distance corresponding to these differentials can be determined only *A posteriori* if one knows the quantities $g_{\mu\nu}$ *A priori*, which determine the gravitational field. This can also be expressed in the following way:

The gravitational field affects the measuring bodies and clocks in a certain way.

From the fundamental equation

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu\,,$$

it can be seen that a further specification is required to determine the physical dimension of the quantities $g_{\mu\nu}$ and x_ν. The quantity $\mathrm{d}s$ has the dimension of length. We also want to regard x_ν as a length (as well as x_4), i.e., we do not assign any physical dimension to the quantities $g_{\mu\nu}$.

5. The Motion of Continuously Distributed Incoherent Masses in an Arbitrary Gravitational Field

In order to derive the law of motion of continuously distributed incoherent masses, we calculate the momentum and the ponderomotive force[1] per unit volume and apply the law of the conservation of momentum to these quantities.

To this end, we must first calculate the three-dimensional volume $\mathscr{V}$ of our mass point. We consider an infinitely small (four-dimensional) piece of the space-time streamline of our material point. Its volume is given by

$$\iiiint \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4 = \mathscr{V}\cdot\mathrm{d}t\,.$$

If we introduce the natural differentials $\mathrm{d}\xi$ in place of the $\mathrm{d}x$, where the measuring body is assumed to be at rest relative to the material point, we have to set

$$\iiint \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 = \mathscr{V}_0$$

i.e., equal to the "resting volume" of the material point.
Further, we have

$$\int \mathrm{d}\xi_4 = \mathrm{d}s\,,$$

where $\mathrm{d}s$ has the same meaning as above.
If the $\mathrm{d}s$ are related to the $\mathrm{d}\xi$ by the substitution

$$\mathrm{d}x_\mu = \sum_\sigma \alpha_{\mu\sigma}\,\mathrm{d}\xi_\sigma\,,$$

we then have the relationship

$$\iiiint \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4 = \iiiint \frac{\partial\,(\mathrm{d}x_1,\mathrm{d}x_2,\mathrm{d}x_3,\mathrm{d}x_4)}{\partial\,(\mathrm{d}\xi_1,\mathrm{d}\xi_2,\mathrm{d}\xi_3,\mathrm{d}\xi_4)}\cdot\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,\mathrm{d}\xi_4$$

or

$$\mathscr{V}\,\mathrm{d}t = \mathscr{V}_0\,\mathrm{d}s\cdot\left|\alpha_{\rho\sigma}\right|\,.$$

[1] For explanation of the term "ponderomotive force" see https://en.wikipedia.org/wiki/Ponderomotive_force

But since

$$\begin{aligned} \mathrm{d}s^2 &= \sum_{\mu\nu} g_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu \\ &= \sum_{\mu\nu\rho\sigma} g_{\mu\nu}\,\alpha_{\mu\rho}\alpha_{\nu\sigma}\,\mathrm{d}\xi_\rho\,\mathrm{d}\xi_\sigma \\ &= \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2 - \mathrm{d}\xi_4^2\,, \end{aligned}$$

there exists the following relation between the determinant

$$g = \left|g_{\mu\nu}\right|\,,$$

i.e., the discriminant of the quadratic differential form $\mathrm{d}s^2$, and the substitutional determinant

$$\left|\alpha_{\rho\sigma}\right|$$

according to

$$\begin{aligned} g\cdot\left(\left|\alpha_{\rho\sigma}\right|\right)^2 &= -1\,, \\ \left|\alpha_{\rho\sigma}\right| &= \frac{1}{\sqrt{-g}}\,. \end{aligned}$$

Thus, one obtains the following relation for $\mathscr{V}$:

$$\mathscr{V}\,\mathrm{d}t = \mathscr{V}_0\,\mathrm{d}s\cdot\frac{1}{\sqrt{-g}}\,.$$

From this one obtains, taking into account Equation 3.3, Equation 3.4, and Equation 3.5 and also substituting ρ_0 for $\frac{m}{\mathscr{V}_0}$, the formulæ

$$\begin{aligned} \frac{J_x}{\mathscr{V}} &= -\rho_0\cdot\sqrt{-g}\cdot\sum_\nu g_{1\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}s}\cdot\frac{\mathrm{d}x_4}{\mathrm{d}s}\,, \\ -\frac{\mathrm{E}}{\mathscr{V}} &= -\rho_0\cdot\sqrt{-g}\cdot\sum_\nu g_{4\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}s}\cdot\frac{\mathrm{d}x_4}{\mathrm{d}s}\,, \\ \frac{\mathcal{F}_x}{\mathscr{V}} &= -\frac{1}{2}\rho_0\cdot\sqrt{-g}\cdot\sum_{\mu\nu}\frac{\partial\, g_{\mu\nu}}{\partial x_1}\cdot\frac{\mathrm{d}x_\mu}{\mathrm{d}s}\cdot\frac{\mathrm{d}x_\nu}{\mathrm{d}s}\,. \end{aligned}$$

We note that

$$\Theta_{\mu\nu} = \rho_0\cdot\frac{\mathrm{d}x_\mu}{\mathrm{d}s}\cdot\frac{\mathrm{d}x_\nu}{\mathrm{d}s}$$

is a second-rank contravariant tensor with respect to arbitrary substitutions. From the above, it can be assumed that the impulse energy conservation law will have the form

$$\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\cdot g_{\sigma\mu}\cdot\Theta_{\mu\nu}\right) - \frac{1}{2}\cdot\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial\, g_{\mu\nu}}{\partial x_\sigma}\cdot\Theta_{\mu\nu} = 0\,. \quad (\sigma = 1,2,3,4)\,. \tag{5.1}$$

The first three of these equations ($\sigma = 1, 2, 3$) express the momentum law, and the last one ($\sigma = 4$) the energy law. It turns out that these equations are in fact covariant with respect to arbitrary substitutions (see section 5.1). Furthermore, the equations of motion of the material point, from which we started, can be derived from the former by integration over the streamline.

We call the tensor $\Theta_{\mu\nu}$ the (contravariant) stress-energy tensor of the material stream. We ascribe to Equation 5.1 a range of validity that extends far beyond the special case of the stream of incoherent masses. The equation represents in general the energy balance between the gravitational field and an arbitrary material process; one has only to substitute for $\Theta_{\mu\nu}$ the stress-energy tensor corresponding to the material system under consideration. The first sum in the equation contains the local derivatives of the stresses or energy flux density, respectively, and the time derivatives of the momentum density or energy density, respectively; the second sum is an expression for the effects exerted by the gravitational field on the material process.

6. The Differential Equations of the Gravitational Field

Having established the momentum-energy equation for material processes (mechanistical, electrical, and other processes) in relation to the gravitational field, there remains for us only the following task. Let the tensor $\Theta_{\mu\nu}$ be given for the material process. Now the question is: "What kind of differential equations enable us to determine the quantities g_{ik}, i.e., the gravitational field?" In other words, we seek the generalization of Poisson's equation or Poisson's equation for gravity in the form

$$\Delta\varphi = 4\pi k \cdot \rho$$

or

$$\nabla^2\phi = 4\pi G \cdot \rho\,,$$

where φ is a real or complex-valued function on a manifold, Δ is the Laplace operator, ∇ is the nabla operator, ϕ is a scalar potential, G and k, respectively, is the gravitational constant and ρ is the object's density.
We have not found a method for solving this problem that is as complete and consistent as the one used to solve the problem discussed earlier. It was necessary to introduce a number of assumptions whose correctness seems plausible but is nevertheless not evident.

The generalization sought will probably be of the form

$$\varkappa \cdot \Theta_{\mu\nu} = \Gamma_{\mu\nu}\,, \tag{6.1}$$

where $\varkappa$ is a constant[1] and $\Gamma_{\mu\nu}$ a second rank contravariant tensor derived from the fundamental tensor $g_{\mu\nu}$ by differential operations. In line with the Newton-Poisson law one would be inclined to require that these Equation 6.1 shall be of second order. But it must be pointed out that, given this assumption, it proves to be impossible to find a differential expression $\Gamma_{\mu\nu}$ that is a generalization of $\Delta\varphi$ and which proves to be a tensor with respect to arbitrary transformations (see section 5.2). *A priori*, however, it cannot be denied that the final, precise equations of gravity could be of a higher order than second order. Therefore there still exists the possibility that the perfectly exact differential equations of gravity could be covariant with respect to arbitrary substitutions. But given the present state of our knowledge about the physical nature of the gravitational field, the attempt to discuss such possibilities would be premature. For this reason we have to confine ourselves to the second order, and we must therefore refrain from formulating gravitational

[1] It turns out that

$$\varkappa = \frac{8\pi G}{c^4}\,,$$

which is called the Einstein constant.

equations that prove to be covariant with respect to arbitrary transformations. Incidentally, it should be emphasized that we have no evidence whatsoever for a general covariance of the gravitational equations[2].

The LAPLACEian scalar $\Delta\varphi$ is obtained from the scalar φ if one forms (i) the expansion (the gradient) of the latter and (ii) the inner operator (the divergence) of the same:

$$\Delta\varphi = \operatorname{div}\operatorname{grad}\varphi = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2}\,.$$

Both operations can be generalized in such a way that one may apply them to every tensor of arbitrarily high rank and allowing for any substitutions of the basic variables (see chapter 3). However, these operations fail when performed on the fundamental tensor $g_{\mu\nu}$[3]. It seems to follow that the equations we are looking for will only be covariant with respect to a certain group of transformations, which is currently unknown to us.
In this situation, considering the old theory of relativity, it seems natural to assume that the linear transformations may be included in the transformation group we are looking for. We therefore require that $\Gamma_{\mu\nu}$ be a tensor with respect to arbitrary linear transformations.

Now it is easy to prove (by performing the transformation) the following theorems:

Theorem 1. *If $\Theta_{\alpha\beta\cdots\lambda}$ is a contravariant tensor of rank n with respect to linear transformations, then*

$$\sum_{\mu} \gamma_{\mu\nu} \cdot \frac{\partial\Theta_{\alpha\beta\cdots\lambda}}{\partial x_\mu}$$

is a contravariant tensor of rank $n + 1$ with respect to linear transformations (Complementation).

$\gamma_{\mu\nu}$ is the contravariant tensor reciprocal to $g_{\mu\nu}$, see chapter 2.

Theorem 2. *If $\Theta_{\alpha\beta\cdots\lambda}$ is a contravariant tensor of rank n with respect to linear transformations, then*

$$\sum_{\lambda} \frac{\partial\Theta_{\alpha\beta\cdots\lambda}}{\partial x_\lambda}$$

is a contravariant tensor of rank $n - 1$ with respect to linear transformations (divergence).

If one carries out these two operations on a tensor in succession, one obtains a tensor of the same rank as the original one (operation Δ, carried out on a tensor). For the fundamental tensor $\gamma_{\mu\nu}$ one obtains

$$\sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta} \cdot \frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\right)\,. \tag{a}$$

One can also see from the following argument that this operator is related to the LAPLACE operator[4].

[2] See also the considerations given at the beginning of chapter 8.
[3] See the note on page 37 in the second part
[4] For an explanation of the LAPLACE operator or LAPLACEian see `https://en.wikipedia.org/wiki/Laplace_operator`

In the theory of relativity (absence of a gravitational field) one would have to set

$$\begin{aligned} g_{11} = g_{22} = g_{33} &= -1\,, \\ g_{44} &= c^2\,, \\ g_{\mu\nu} &= 0\,, \\ \text{for } \mu &\neq \nu\,; \end{aligned}$$

therefore

$$\begin{aligned} \gamma_{11} = \gamma_{22} = \gamma_{33} &= -1\,, \\ \gamma_{44} &= \frac{1}{c^2}\,, \\ \gamma_{\mu\nu} &= 0\,, \\ \text{for } \mu &\neq \nu\,. \end{aligned}$$

If a gravitational field is present that is sufficiently weak, i.e., if the $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ differ only infinitesimally from the values just given, then one obtains instead of Equation a, neglecting the second-order terms

$$-\left(\frac{\partial^2\gamma_{\mu\nu}}{\partial x_1^2} + \frac{\partial^2\gamma_{\mu\nu}}{\partial x_2^2} + \frac{\partial^2\gamma_{\mu\nu}}{\partial x_3^2} - \frac{1}{c^2}\cdot\frac{\partial^2\gamma_{\mu\nu}}{\partial x_4^2}\right).$$

If the field is a static one and only g_{44} is variable, then we finally arrive at Newton's theory of gravitation if we set the just formed expression equal too $\Gamma_{\mu\nu}$ except a constant.

Hence one might think that, up to a constant factor, Equation a must already be the generalization of $\Delta\varphi$ that we are looking for. But this would be a mistake; because alongside this expression there may also appear other terms in such a specific generalization which are themselves tensors and which vanish if we perform the same kind of neglections just indicated. This always occurs when two first derivatives of the $g_{\mu\nu}$ or $\gamma_{\mu\nu}$ appear to be multiplied with each other. So, for example,

$$\sum_{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\mu}\cdot\frac{\partial\gamma_{\alpha\beta}}{\partial x_\nu}$$

is a covariant tensor of rank two (with respect to linear transformations); This tensor becomes infinitely small with respect to second-order derivatives if the quantities $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ of constants differ only by infinitely small ones from the first-order derivatives. We must therefore allow other terms to appear in $\Gamma_{\mu\nu}$ alongside Equation a, which for the time being only need to satisfy the condition that, taken together, they must have a tensor character with respect to linear transformations.

To find these terms we make use of the momentum-energy theorem. To make the method used clear, I will first apply it to a well-known example.
In electrostatics $-\frac{\partial\varphi}{\partial x_\nu}\cdot\rho$ is the ν-th component of the momentum transferred to the matter per unit volume, if φ denotes the electrostatic potential and ρ the electrical density (charge density). We wish to find a differential equation for φ of such kind that the law of the conservation of

momentum is always fulfilled. It is well known that the equation

$$\sum_{\nu} \frac{\partial^2 \varphi}{\partial x_\nu^2} = \rho$$

solves this problem. The fact that the law of conservation of momentum is satisfied follows from the identity

$$\begin{aligned} &\sum_{\mu} \frac{\partial}{\partial x_\mu} \left(\frac{\partial \varphi}{\partial x_\nu} \frac{\partial \varphi}{\partial x_\mu} \right) - \frac{\partial}{\partial x_\nu} \left(\frac{1}{2} \cdot \sum_{\mu} \left(\frac{\partial \varphi}{\partial x_\mu} \right)^2 \right) \\ &= \frac{\partial}{\partial x_\nu} \sum_{\mu} \frac{\partial^2 \varphi}{\partial x_\mu^2} \left(= - \frac{\partial \varphi}{\partial x_\nu} \cdot \rho \right) . \end{aligned}$$

So if the conservation of momentum is satisfied, there must be an identical equation with the following structure for every ν: On the right-hand side is $-\frac{\partial \varphi}{\partial x_\nu}$ multiplied by the left-hand side of the differential equation. On the left side of this identity is a sum of differential quotients.

If the differential equation for φ were not yet known, the problem of finding it could be reduced to that of finding the identical equation mentioned above. However, it is important for us to note that this identity can be derived if one of the terms appearing in it is known. All we have to do is repeatedly apply the rule for differentiating a product, i.e. the Leibniz product rule in the forms

$$\frac{\partial}{\partial x_\nu}(uv) = \frac{\partial u}{\partial x_\nu} \cdot v + \frac{\partial v}{\partial x_\nu} \cdot u$$

and

$$u \cdot \frac{\partial v}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}(uv) - \frac{\partial u}{\partial x_\nu} \cdot v$$

and finally move the terms that are differential quotients to the left side of the equation and the remaining terms to the right side. For example, if one starts with the first term of the above identity, one obtains in sequence

$$\begin{aligned} \sum_{\mu} \frac{\partial}{\partial x_\mu} \left(\frac{\partial \varphi}{\partial x_\nu} \frac{\partial \varphi}{\partial x_\mu} \right) &= \sum_{\mu} \frac{\partial \varphi}{\partial x_\nu} \cdot \frac{\partial^2 \varphi}{\partial x_\mu^2} \\ &\quad + \sum_{\mu} \frac{\partial \varphi}{\partial x_\mu} \cdot \frac{\partial^2 \varphi}{\partial x_\nu \partial x_\mu} \\ &= \frac{\partial \varphi}{\partial x_\nu} \cdot \sum_{\mu} \frac{\partial^2 \varphi}{\partial x_\mu^2} + \frac{\partial}{\partial x_\nu} \left\{ \frac{1}{2} \cdot \sum_{\mu} \left(\frac{\partial \varphi}{\partial x_\mu} \right)^2 \right\} , \end{aligned}$$

from which we obtain the above identity upon rearrangement.

We now turn again to our problem. It follows from Equation 5.1 that

$$\frac{1}{2} \cdot \sum_{\mu\nu} \sqrt{-g} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \Theta_{\mu\nu}\,,$$

is the momentum (or energy) transferred by the gravitational field to the matter per unit volume. In order for the law of conservation of energy and momentum to be satisfied, the differential expressions $\Gamma_{\mu\nu}$ of the fundamental quantities $\gamma_{\mu\nu}$, which are included in the gravitational equations

$$\varkappa \cdot \Theta_{\mu\nu} = \Gamma_{\mu\nu}\,,$$

must be chosen such that

$$\frac{1}{2\varkappa} \cdot \sum_{\mu\nu} \sqrt{-g} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \Gamma_{\mu\nu}$$

can be transformed in such a way that the law of conservation of energy and momentum appears as the sum of differential quotients. On the other hand, it is known that Equation a appears in the expression sought for $\Gamma_{\mu\nu}$. Hence the identical equation described above which is being sought has the following form:

$$\begin{aligned}&\text{Sum of differential quotients}\\ &= \frac{1}{2\varkappa} \cdot \sum_{\mu\nu} \sqrt{-g} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \left\{ \sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha} \left(\gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) \right.\\ &\left. +\ \text{other terms, which vanish with the first approximation.} \right\}\end{aligned}$$

The identity that is being sought is thereby uniquely determined; if one constructs it according to the procedure indicated (see section 5.3) one obtains

$$\left.\begin{aligned}&\sum_{\alpha\beta\tau\rho} \frac{\partial}{\partial x_\alpha} \left(\sqrt{-g} \cdot \gamma_{\alpha\beta} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} \cdot \frac{\partial g_{\tau\rho}}{\partial x_\sigma} \right)\\ &- \frac{1}{2} \cdot \sum_{\alpha\beta\tau\rho} \frac{\partial}{\partial x_\sigma} \left(\sqrt{-g} \cdot \gamma_{\alpha\beta} \frac{\partial \gamma_{\tau\rho}}{\partial x_\alpha} \cdot \frac{\partial g_{\tau\rho}}{\partial x_\beta} \right)\\ &= \sum_{\mu\nu} \sqrt{-g} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \left\{ \sum_{\alpha\beta} \frac{1}{\sqrt{-g}} \cdot \frac{\partial}{\partial x_\alpha} \left(\gamma_{\alpha\beta} \cdot \sqrt{-g} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) \right.\\ &- \sum_{\alpha\beta\tau\rho} \gamma_{\alpha\beta} g_{\tau\rho} \cdot \frac{\partial \gamma_{\mu\tau}}{\partial x_\alpha} \cdot \frac{\partial \gamma_{\nu\rho}}{\partial x_\beta}\\ &\left. + \frac{1}{2} \cdot \sum_{\alpha\beta\tau\rho} \gamma_{\alpha\mu} \gamma_{\beta\nu} \cdot \frac{\partial g_{\tau\rho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} - \frac{1}{4} \cdot \sum_{\alpha\beta\tau\rho} \gamma_{\mu\nu} \gamma_{\alpha\beta} \cdot \frac{\partial g_{\tau\rho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} \right\}.\end{aligned}\right\} \tag{6.2}$$

Thus, the expression for $\Gamma_{\mu\nu}$ which is enclosed between the curly brackets on the right-hand side is the tensor we were looking for. This tensor enters into the gravitational equations according to

$$\varkappa \cdot \Theta_{\mu\nu} = \Gamma_{\mu\nu}$$

To make these equations easier to understand, we introduce the following abbreviations:

$$-2\varkappa \cdot \vartheta_{\mu\nu} = \sum_{\alpha\beta\tau\rho} \left(\gamma_{\alpha\mu}\gamma_{\beta\nu} \cdot \frac{\partial g_{\tau\rho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} - \frac{1}{2} \cdot \gamma_{\mu\nu}\gamma_{\alpha\beta} \cdot \frac{\partial g_{\tau\rho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} \right) . \tag{6.3}$$

We will designate $\vartheta_{\mu\nu}$ as the "*contravariant stress-energy tensor of the gravitational field*". The covariant tensor reciprocal to $\vartheta_{\mu\nu}$ will be denoted by $t_{\mu\nu}$; then we have

$$-2\varkappa \cdot t_{\mu\nu} = \sum_{\alpha\beta\tau\rho} \left(\frac{\partial g_{\tau\rho}}{\partial x_\mu} \frac{\partial \gamma_{\tau\rho}}{\partial x_\nu} - \frac{1}{2} \cdot g_{\mu\nu}\, \gamma_{\alpha\beta} \cdot \frac{\partial g_{\tau\rho}}{\partial x_\alpha} \frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} \right) . \tag{6.4}$$

Likewise, for the sake of simplification, we introduce the following notations for differential operations applied to the fundamental tensors γ and g:

$$\left.\begin{aligned} \Delta_{\mu\nu}(\gamma) = &\sum_{\alpha\beta} \frac{1}{\sqrt{-g}} \cdot \frac{\partial}{\partial x_\alpha} \left(\gamma_{\alpha\beta} \cdot \sqrt{-g} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) \\ &- \sum_{\alpha\beta\tau\rho} \gamma_{\alpha\beta} g_{\tau\rho} \cdot \frac{\partial \gamma_{\mu\tau}}{\partial x_\alpha} \frac{\partial \gamma_{\nu\rho}}{\partial x_\beta} , \end{aligned}\right\} \tag{6.5}$$

and

$$\left.\begin{aligned} D_{\mu\nu}(g) = &\sum_{\alpha\beta} \frac{1}{\sqrt{-g}} \cdot \frac{\partial}{\partial x_\alpha} \left(\gamma_{\alpha\beta} \cdot \sqrt{-g} \cdot \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right) \\ &- \sum_{\alpha\beta\tau\rho} \gamma_{\alpha\beta}\gamma_{\tau\rho} \cdot \frac{\partial g_{\mu\tau}}{\partial x_\alpha} \frac{\partial g_{\nu\rho}}{\partial x_\beta} , \end{aligned}\right\} \tag{6.6}$$

respectively. Each of these operators results again in a tensor of the same kind (with respect to linear transformations).

With the application of these abbreviations the identity according to 6.2 takes the form

$$\left.\begin{aligned} \sum_{\mu\nu} \frac{\partial}{\partial x_\nu} \left\{ \sqrt{-g} \cdot g_{\sigma\mu} \cdot \varkappa\vartheta_{\mu\nu} \right\} = &\frac{1}{2} \cdot \sum_{\mu\nu} \sqrt{-g} \\ &\cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \left\{ -\Delta_{\mu\nu}(\gamma) + \varkappa\vartheta_{\mu\nu} \right\} , \end{aligned}\right\} \tag{6.2a}$$

or likewise

$$\left.\begin{aligned}\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}\left\{\sqrt{-g}\cdot\gamma_{\mu\nu}\cdot\varkappa t_{\mu\nu}\right\} &= \frac{1}{2}\cdot\sum_{\mu\nu}\sqrt{-g}\\ &\quad\cdot\frac{\partial\, g_{\mu\nu}}{\partial x_\sigma}\left\{-\Delta_{\mu\nu}(\gamma)+\varkappa\vartheta_{\mu\nu}\right\}\,.\end{aligned}\right\} \tag{6.2b}$$

If we rewrite both laws, the law of conservation according to Equation 5.1 for matter and the law of conservation according to Equation 6.2a for the gravitational field in the form

$$\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\cdot g_{\sigma\mu}\cdot\Theta_{\mu\nu}\right)-\frac{1}{2}\cdot\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial\, g_{\mu\nu}}{\partial x_\sigma}\cdot\Theta_{\mu\nu}=0 \tag{5.1'}$$

$$\left.\begin{aligned}&\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\cdot g_{\sigma\mu}\cdot\vartheta_{\mu\nu}\right)-\frac{1}{2}\cdot\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial\, g_{\mu\nu}}{\partial x_\mu}\cdot\vartheta_{\mu\nu}\\ &=-\frac{1}{2\varkappa}\cdot\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial\, g_{\mu\nu}}{\partial x_\sigma}\cdot\Delta_{\mu\nu}(\gamma)\,,\end{aligned}\right\} \tag{6.2c}$$

then one recognizes that the stress-energy tensor $\vartheta_{\mu\nu}$ of the gravitational field enters the law of conservation for the gravitational field in exactly the same way as the tensor $\Theta_{\mu\nu}$ of the material process enters the law of conservation for this process; this is a remarkable circumstance considering the difference in the derivation of the two laws.

From Equation 6.2a follows the expression for the differential tensor entering into the gravitational equations according to

$$\Gamma_{\mu\nu}=\Delta_{\mu\nu}(\gamma)-\varkappa\cdot\vartheta_{\mu\nu}\,. \tag{6.7}$$

Thus, the gravitational equations according to Equation 6.1 are of the form

$$\Delta_{\mu\nu}(\gamma)=\varkappa\cdot\left(\Theta_{\mu\nu}+\vartheta_{\mu\nu}\right) \tag{6.8}$$

These equations satisfy a requirement that, in our opinion, must necessarily be imposed on a theory of relativity of gravitation; that is to say, these equations show that the tensor $\vartheta_{\mu\nu}$ of the gravitational field acts in a same field-generating manner as the tensor $\Theta_{\mu\nu}$ of the material processes. An exceptional position of gravitational energy in comparison to all other kinds of energies would lead to untenable consequences.

Adding Equation 5.1 and Equation 6.2a while taking into account Equation 6.8, one finds

$$\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}\left\{\sqrt{-g}\cdot g_{\sigma\mu}\cdot\left(\Theta_{\mu\nu}+\vartheta_{\mu\nu}\right)\right\}=0\,. \tag{6.9}$$

We therefore state the following:

Lemma 1. *The procedure just described shows that the conservation laws apply to both entities, matter and gravitational field.*

In the foregoing we have given preference to the contravariant tensors, because the contravariant stress-energy tensor of the stream of incoherent masses can be expressed in an especially simple manner. However, we can express the fundamental relations that we have obtained just as simply by using *covariant* tensors. Instead of $\Theta_{\mu\nu}$, we then have to set

$$T_{\mu\nu} = \sum_{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} \Theta_{\alpha\beta}$$

as the *covariant* stress-energy tensor of the material process[5]. Instead of Equation 5.1, we obtain through term-by-term rearranging

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu} \left(\sqrt{-g} \cdot \gamma_{\sigma\nu} T_{\mu\nu} \right) + \frac{1}{2} \cdot \sum_{\mu\nu} \sqrt{-g} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma} \cdot T_{\mu\nu} = 0 \,. \tag{6.10}$$

It follows from this Equation 6.10 and Equation 6.6 that the equations of the gravitational field can also be written in the form

$$-D_{\mu\nu}(g) = \varkappa \cdot \left(t_{\mu\nu} + T_{\mu\nu} \right) \tag{6.11}$$

and which can also be derived directly from Equation 6.8. Analogously to Equation 6.9 the following relationship holds:

$$\sum_{\nu} \frac{\partial}{\partial x_\nu} \left\{ \sqrt{-g} \cdot \gamma_{\sigma\mu} \cdot \left(T_{\mu\nu} + t_{\mu\nu} \right) \right\} = 0 \,. \tag{6.12}$$

[5]For a modern explanation in also modern notation see `https://en.wikipedia.org/wiki/Stress%E2%80%93energy_tensor`, section *Covariant and mixed forms*

7. Influence of the Gravitational Field on Physical Events, Especially on Electromagnetic Actions

Since momentum and energy play a role in every physical process and, for their part, also determine the gravitational field and are influenced by it, the variables $g_{\mu\nu}$ that determine the gravitational field must be included in all physical equation systems. Thus, we have seen that the motion of the material point is determined by the equation

$$\delta\left\{\int \mathrm{d}s\right\} = 0\,,$$

where

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu$$

is an invariant with respect to arbitrary substitutions ($\mathrm{d}s^2$ is called the *line element*.). The equations we are looking for, which determine the course of any physical process, must now be such that the invariance of $\mathrm{d}s$ results in the covariance of the relevant system of equations.

However, when discussing these general tasks, we first encounter a fundamental difficulty. We do not know with respect to which group of transformations the equations we are looking for must be covariant. At first glance, it seems most natural to require that the systems of equations be covariant with respect to arbitrary transformations. However, this is contradicted by the fact that the equations of the gravitational field we have established do not possess this property. We have only been able to prove that the gravitational equations are covariant with respect to arbitrary linear transformations; however, we do not know whether there is a general transformation group with respect to which these equations are covariant. The question of the existence of such a group for the system of equations Equation 6.8 or Equation 6.11 is the most important question of all that follows from the explanations given here. In any case, at the current state of the theory, we are not justified in demanding the covariance of physical equations with respect to arbitrary substitutions.

On the other hand, however, we have seen that it was possible to establish an energy-momentum balance equation for material processes (cf. chapter 5, Equation 5.1) that allows for arbitrary transformations. It therefore seems natural to assume that all physical systems of equations, with the exception of the gravitational equations, must be formulated in such a way that they are covariant with respect to arbitrary substitutions. In my opinion, the exceptional position of the gravitational equations in this respect compared to all other systems is related to the fact that only the former equations are allowed to contain the second derivatives of the components of the fundamental tensor.

The formulation of such systems of equations requires the tools of generalized vector analysis, as described in Part I.

In this case, we will limit ourselves to explaining how to obtain the electromagnetic field equations for a vacuum in this way[1]. We assume that the electric charge is to be regarded as something unchangeable. An infinitely small, arbitrarily moving body is charged with the charge e and, for a moving body, it has the volume $\mathrm{d}\mathscr{V}_0$ (rest volume). We define $\frac{e}{\mathrm{d}\mathscr{V}_0} = \rho_0$ as the true density of electricity; according to the definition, this is a scalar. It is therefore

$$\rho_0 \cdot \frac{\mathrm{d}x_\nu}{\mathrm{d}s}$$

a contravariant four-vector, which we now transform by defining the density ρ of electricity in relation to the coordinate system by the equation

$$\rho_0 \cdot \mathrm{d}\mathscr{V}_0 = \rho \cdot \mathrm{d}\mathscr{V} \ .$$

Using the equation

$$\mathrm{d}\mathscr{V}_0 \, \mathrm{d}s = \sqrt{-g} \cdot \mathrm{d}\mathscr{V} \cdot \mathrm{d}t$$

from chapter 5 we obtain

$$\rho_0 \cdot \frac{\mathrm{d}v_\nu}{\mathrm{d}s} = \frac{1}{\sqrt{-g}} \cdot \rho \cdot \frac{\mathrm{d}x_\nu}{\mathrm{d}t} \, ,$$

i.e., the contravariant vector of the electrical stream.

We reduce the electromagnetic field to a special, contravariant tensor of second rank $\varphi_{\mu\nu}$ (a six-vector), and form the "dual" contravariant tensor of second rank $\varphi^\star_{\mu\nu}$ by the method explained in chapter 4, Equation 4.7. According to Equation 4.5 of Part II, the divergence of a special contravariant tensor of second rank is

$$\frac{1}{\sqrt{-g}} \cdot \sum_\nu \frac{\partial}{\partial x_\nu} \left(\sqrt{-g} \cdot \varphi_{\mu\nu} \right) \ .$$

As a generalization of the Maxwell-Lorentz field equations, we set up the equations

$$\sum_\nu \frac{\partial}{\partial x_\nu} \left(\sqrt{-g} \cdot \varphi_{\mu\nu} \right) = \rho \cdot \frac{\mathrm{d}x_\mu}{\mathrm{d}t} \, , \tag{7.1}$$

[1] See also the treatise by Kottler cited below on page 32.

$$\sum_{\nu} \frac{\partial}{\partial x_\nu} \left(\sqrt{-g} \cdot \varphi^{\star}_{\mu\nu} \right) = 0 \,, \tag{7.2}$$

the covariance of which is self-evident. If we set

$$\begin{aligned} \sqrt{-g} \cdot \varphi_{23} &= \mathfrak{H}_x \,, \\ \sqrt{-g} \cdot \varphi_{31} &= \mathfrak{H}_y \,, \\ \sqrt{-g} \cdot \varphi_{12} &= \mathfrak{H}_z \,; \end{aligned}$$

$$\begin{aligned} \sqrt{-g} \cdot \varphi_{14} &= -\mathfrak{G}_x \,, \\ \sqrt{-g} \cdot \varphi_{24} &= -\mathfrak{G}_y \,, \\ \sqrt{-g} \cdot \varphi_{34} &= -\mathfrak{G}_z \,, \end{aligned}$$

and

$$\rho \cdot \frac{\mathrm{d}x_\mu}{\mathrm{d}t} = u_\mu \,,$$

then the system of equations according to Equation 7.1, written out in a more detailed manner, takes the form

$$\begin{aligned} \frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z} - \frac{\partial \mathfrak{H}_x}{\partial t} &= u_x \\ \cdots \quad \cdots \quad \cdots \quad &\vdots \\ \frac{\partial \mathfrak{G}_x}{\partial x} + \frac{\partial \mathfrak{G}_y}{\partial y} + \frac{\partial \mathfrak{G}_z}{\partial z} &= \rho \,, \end{aligned}$$

Up to the choice of the units, these equations coincide with Maxwell's first system. To form the second system, it should first be noted that to the components

$$\begin{aligned} &\mathfrak{H}_x, \quad \mathfrak{H}_y, \quad \mathfrak{H}_z, \\ &-\mathfrak{G}_x, \quad -\mathfrak{G}_y, \quad -\mathfrak{G}_z \end{aligned}$$

of

$$\sqrt{-g} \cdot \varphi_{\mu\nu} \,,$$

there belong the components

$$\begin{aligned} &-\mathfrak{G}_x, \quad -\mathfrak{G}_y, \quad -\mathfrak{G}_z, \\ &\mathfrak{H}_x, \quad \mathfrak{H}_y, \quad \mathfrak{H}_z, \end{aligned}$$

of the complement $f_{\mu\nu}$ (see chapter 4, Equation 4.6a). In the absence of a gravitational field, this

results in the second system, i.e., Equation 7.2 in the form

$$\begin{aligned} -\frac{\partial \mathfrak{G}_z}{\partial x} + \frac{\partial \mathfrak{G}_y}{\partial z} - \frac{1}{c^2} \cdot \frac{\partial \mathfrak{G}_x}{\partial t} &= 0 \\ \cdots \quad \cdots \quad \cdots \quad &\vdots \\ -\frac{1}{c^2} \cdot \frac{\partial \mathfrak{H}_x}{\partial x} - \frac{1}{c^2} \cdot \frac{\partial \mathfrak{H}_y}{\partial t} - \frac{1}{c^2} \cdot \frac{\partial \mathfrak{H}_z}{\partial z} &= 0\,. \end{aligned}$$

This proves that the equations we have established truly constitute a generalization of those of ordinary relativity theory.

8. Is it Possible to Reduce the Gravitational Field to a Scalar?

Given the undeniable complexity of the theory of gravity presented here, we must seriously ask ourselves whether the view that has been exclusively held to date, according to which the gravitational field can be reduced to a scalar Φ, is the only obvious and justified one. I would like to briefly explain why we believe this question must be answered in the negative.
Characterizing the gravitational field by means of a scalar opens up a path that is entirely analogous to the one taken above. The equation

$$\delta \left\{ \int \Phi \, \mathrm{d}s \right\} = 0$$

is used as the equation of motion of the material point in HAMILTON's form, where $\mathrm{d}s$ is the 4-dimensional line element of ordinary relativity theory and Φ is a scalar. We then proceed in exactly the same way as before, without having to leave ordinary relativity theory.
Here, too, the material process of any kind is characterized by a stress-energy tensor $T_{\mu\nu}$. In this view, however, a scalar is decisive for the interaction between the gravitational field and the material process. As Mr. LAUE pointed out to me, this scalar can only be

$$\sum_{\mu} T_{\mu\nu} = P \,,$$

which I would like to refer to as LAUE's scalar (see chapter 2, last formula). Then the theorem of the equivalence of inertial and gravitational mass can also be satisfied to a certain extent here. Mr. LAUE pointed out to me that for a closed system,

$$\int P \, \mathrm{d}\mathscr{V} = \int T_{44} \, \mathrm{d}\tau \,.$$

From this it is clear that, even according to this view, the total energy of a closed system is decisive for its gravity. However, the gravity of open systems would depend on the orthogonal stresses T_{11}, etc. to which the system is subjected. This has consequences that are unacceptable to me, as will be shown by the example of cavity radiation.

It is well known that the scalar P vanishes for radiation in a vacuum. If the radiation is enclosed in a massless, mirrored box, its walls experience tensile stresses that cause the system, viewed as a whole, to acquire a heavy mass $\int P \, \mathrm{d}\tau$, which corresponds to the energy E of the radiation. However, instead of enclosing the radiation in a hollow box, I imagine the radiation to be bounded by (i) the mirrored walls of a fixed shaft S, and (ii) two vertically movable, mirrored walls W_1 and W_2, which are firmly connected to each other by a rod according to Figure 8.1.

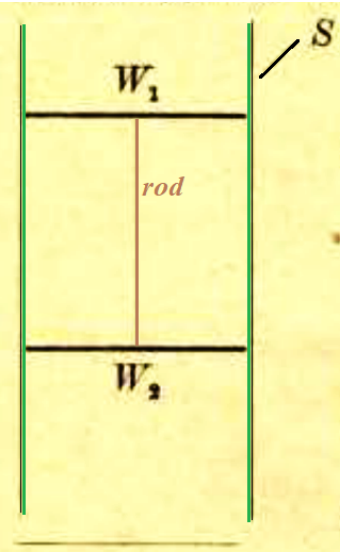

Figure 8.1.: Fixed, mirrored shaft containing radiation

In this case, the gravitational mass $\int P \, d\tau$ of the moving system is only one third of the value that occurs with a box that moves as a whole. This means that only one third of the work would be required to lift the radiation against a gravitational field, in contrast to the case considered earlier, where the radiation is enclosed in a box. This seems unacceptable to me. I must admit, of course, that for me the strongest argument for rejecting such a theory is based on the conviction that relativity is valid not only for orthogonal linear substitutions, but for a much broader substitution group. But we are not entitled to invoke this argument because we have not even been able to identify the (most general) substitution group that belongs to our gravitational equations.

Part II.

Mathematical Part

by Marcel Grossman

1. Introduction – General Remarks

The mathematical tools for developing the vector analysis of a gravitational field, which is characterized by the invariance of the line element

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\, \mathrm{d}x_\mu\, \mathrm{d}x_\nu$$

derive from Christoffel's fundamental paper on the transformation of quadratic differential forms, [2]. Based on Christoffel's results, Ricci and Levi-Civita, [12] developed their methods of absolute differential calculus, i.e., differential calculus independent of the coordinate system, which allow the differential equations of mathematical physics to be given an invariant form. However, since the vector analysis of Euclidean space based on arbitrary curvilinear coordinates is formally identical to the vector analysis of any manifold given by its line element, it is not difficult to extend the vector-analytical concepts developed in recent years by Minkowski, Sommerfeld, Laue, and others for the theory of relativity, to Einstein's general theory described above. With a little practice, the general vector analysis obtained in this way proves to be just as easy to handle as the special vector analysis of three- or four-dimensional Euclidean space; the greater generality of its concepts even gives it a clarity that is often not found in the special case.

The theory of special tensors (see chapter 4) has been dealt with comprehensively in a treatise by Kottler, [8], published during the writing of this work. This was done on the basis of the theory of integral forms, which is not possible in the general case. Since Einstein's theory of gravity, and in particular the problem of the differential equations of the gravitational field, will require more detailed mathematical investigation, a systematic presentation of general vector analysis may be appropriate here. I have deliberately avoided using geometric aids, as I believe they contribute little to the illustration of the concepts of vector analysis.

2. General Tensors

Let

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu \tag{2.1}$$

be the square of the line element, which is viewed as the invariant measure of the distance between two infinitely close space-time points. Except where otherwise noted, the following derivations are independent of the number of variables; let us denote this number by n.

With a transformation according to

$$x_i = x_i\left(x'_1, x'_2, \ldots, x'_n\right) \qquad (i = 1, 2, \ldots, n) \tag{2.2}$$

of the variables, or a transformation according to

$$\left.\begin{aligned} \mathrm{d}x_i &= \sum_k \frac{\partial x_i}{\partial x'_k}\,\mathrm{d}x'_k = \sum_k p_{ik}\,\mathrm{d}x'_k \\ \mathrm{d}x'_i &= \sum_k \frac{\partial x'_i}{\partial x_k}\,\mathrm{d}x_k = \sum_k \pi_{ki}\,\mathrm{d}x_k \end{aligned}\right\} \tag{2.3}$$

of their differentials, the coefficients of the line element transform according to the formulæ

$$g'_{rs} = \sum_{\mu\nu} p_{\mu r} p_{\nu s}\, g_{\mu\nu}\ . \tag{2.4}$$

Let g be the *discriminant* of the differential form Equation 2.1, i.e., the determinant

$$g = \left|g_{\mu\nu}\right|\ .$$

If $\gamma_{\mu\nu}$ is the subdeterminant of g divided by the discriminant ("normalized") that is adjoint to the element $g_{\mu\nu}$, then the quantities $\gamma_{\mu\nu}$ transform according to the formulas formulæ

$$\gamma'_{rs} = \sum_{\mu\nu} \pi_{\mu r}\pi_{\nu s}\gamma_{\mu\nu}\ . \tag{2.5}$$

We now introduce the following definitions:

Definition 1 (covariant tensor)**.** *The sum of a system of functions $T_{i_1 i_2 \ldots i_\lambda}$ of the variables x is called a **covariant tensor of rank** λ if these quantities transform according to the formulæ*

$$T'_{r_1 r_2 \ldots r_\lambda} = \sum_{i_1 i_2 \ldots i_\lambda} p_{i_1 r_1} p_{i_2 r_2} p_{i_3 r_3} \ldots p_{i_\lambda r_\lambda} \cdot T_{i_1 i_2 \ldots i_\lambda} \,. \tag{2.6}$$

Definition 2 (contravariant tensor)**.** *The sum of a system of functions $\Theta_{i_1 i_2 \ldots i_\lambda}$ of the variables x is called a **contravariant tensor of rank** λ if these quantities transform according to the formulæ*[1]

$$\Theta'_{r_1 r_2 \ldots r_\lambda} = \sum_{i_1 i_2 \ldots i_\lambda} \pi_{i_1 r_1} \pi_{i_2 r_2} \pi_{i_3 r_3} \ldots \pi_{i_\lambda r_\lambda} \cdot \Theta_{i_1 i_2 \ldots i_\lambda} \,. \tag{2.7}$$

Definition 3 (mixed tensor)**.** *The sum of a system of functions $\mathfrak{T}_{i_1 i_2 \ldots i_\mu / k_1 k_2 \ldots k_\nu}$ of the variables x is called a **mixed tensor**, covariant of rank μ and contravariant of rank ν if these quantities transform according to the formulæ*

$$\mathfrak{T}'_{r_1 r_2 \ldots r_\mu / s_1 s_2 \ldots s_\nu} = \sum_{\substack{i_1 i_2 \ldots i_\mu \\ k_1 k_2 \ldots k_\nu}} p_{i_1 r_1} p_{i_2 r_2} \ldots p_{i_\mu r_\nu} \cdot \pi_{k_1 s_1} \pi_{k_2 s_2} \ldots \pi_{k_\mu s_\nu} \cdot \mathfrak{T}_{i_1 i_2 \ldots i_\mu / k_1 k_2 \ldots k_\nu} \,. \tag{2.8}$$

From these definitions and Equation 2.4 and Equation 2.5 it follows that: The quantities $g_{\mu\nu}$ form a covariant tensor of second rank, and the quantities $\gamma_{\mu\nu}$ a contravariant tensor also of second rank; in the case $n = 4$, they form the *fundamental tensors of the gravitational field.* According to Equation 2.3, the quantities $\mathrm{d}x_i$ form a contravariant tensor of rank one. Tensors of rank one are also called vectors of the first kind or four-vectors for $n = 4$.

The following algebraic tensor operations follow immediately from the definitions of a tensor:

Corollary 1. *The sum of two tensors of the same kind of rank λ is again a tensor of the same kind of rank λ, the components of which are formed by the addition of the corresponding components of the two tensors.*

Corollary 2. *The outer product of two covariant (contravariant) tensors of rank λ or μ, respectively, is a covariant (contravariant) tensor of rank $\lambda + \mu$ with the components*

$$T_{i_1 i_2 \ldots i_\lambda k_1 k_2 k_\mu} = A_{i_1 i_2 \ldots i_\lambda} \cdot B_{k_1 k_2 \ldots k_\mu} \tag{2.9}$$

or,

$$\Theta_{i_1 i_2 \ldots i_\lambda k_1 k_2 k_\mu} = \Phi_{i_1 i_2 \ldots i_\lambda} \cdot \Psi_{k_1 k_2 \ldots k_\mu} \,, \tag{2.9'}$$

respectively.

[1] Thus, our covariant (contravariant) tensors of rank λ are identical with the "covariant (contravariant) systems of λ-th order" of Ricci and Levi-Civita and are denoted by these authors by $X_{r_1 r_2 \ldots r_\lambda}$ and $X^{r_1 r_2 \ldots r_\lambda}$, respectively. However as many advantages the latter notation may offer, complications in more complex equations have, nevertheless, forced us to choose the above notation and therefore to denote covariant tensors by Latin letters, contravariant tensors by Greek letters, and mixed tensors by Gothic letters. Covariant and contravariant tensors are special cases of the mixed tensors.

Corollary 3. *We designate as the inner product of two tensors*
(a) the covariant tensor

$$T_{i_1 i_2 \ldots i_\lambda} = \sum_{k_1 k_2 \ldots k_\mu} \Phi_{k_1 k_2 \ldots k_\mu} \cdot A_{i_1 i_2 \ldots i_\lambda k_1 k_2 \ldots k_\mu} , \tag{2.10}$$

b) the contravariant tensor

$$\Theta_{i_1 i_2 \ldots i_\lambda} = \sum_{k_1 k_2 \ldots k_\mu} A_{k_1 k_2 \ldots k_\mu} \cdot \Phi_{i_1 i_2 \ldots i_\lambda k_1 k_2 \ldots k_\mu} , \tag{2.11}$$

(c) the mixed tensor

$$\mathfrak{T}_{r_1 r_2 \ldots r_\mu / s_1 s_2 \ldots s_\nu} = \sum_{k_1 k_2 \ldots k_\lambda} A_{k_1 k_2 \ldots k_\lambda r_1 r_2 \ldots r_\mu} \cdot \Phi_{k_1 k_2 \ldots k_\lambda s_1 s_2 \ldots s_\nu} , \tag{2.12}$$

or, in general, subsuming the three cases (a) to (c) to form (d)

$$\mathfrak{T}_{r_1 r_2 \ldots r_\mu / u_1 u_2 \ldots u_\alpha / s_1 s_2 \ldots s_\nu / v_1 v_2 \ldots v_\beta} = \sum_{k_1 k_2 \ldots k_\lambda} \mathfrak{U}_{r_1 r_2 \ldots r_\mu / k_1 k_2 \ldots k_\lambda / v_1 v_2 \ldots v_\beta} \cdot \mathfrak{B}_{k_1 k_2 \ldots k_\lambda / u_1 u_2 \ldots u_\alpha / s_1 s_2 \ldots s_\nu} . \tag{d}$$

The terms “outer product” and “inner product”, taken from ordinary vector analysis, are justified because those operations ultimately turn out to be special cases of the ones considered here. If, in cases a) or b), the rank λ is zero, then the inner product is a scalar.

Reciprocity of a covariant and a contravariant tensor.

Corollary 4. *From a covariant tensor of rank λ, the reciprocal contravariant tensor of rank λ is formed by λ-fold inner multiplication with the contravariant fundamental tensor according to*

$$\Theta_{i_1 i_2 \ldots i_\lambda} = \sum_{k_1 k_2 \ldots k_\lambda} \gamma_{i_1 k_1} \gamma_{i_2 k_2} \gamma_{i_3 k_3} \ldots \gamma_{i_\lambda k_\lambda} \cdot T_{k_1 k_2 \ldots k_\lambda} , \tag{2.13}$$

from which

$$T_{i_1 i_2 \ldots i_\lambda} = \sum_{k_1 k_2 \ldots k_\lambda} g_{i_1 k_1} g_{i_2 k_2} g_{i_3 k_3} \ldots g_{i_\lambda k_\lambda} \cdot \Theta_{k_1 k_2 \ldots k_\lambda} \tag{2.14}$$

is obtained by resolution.

Hence, one obtains a scalar from a tensor by multiplying the latter with its reciprocal tensor according to the formulæ

$$\sum_{i_1 i_2 \ldots i_\lambda} T_{i_1 i_2 \ldots i_\lambda} \cdot \Theta_{i_1 i_2 \ldots i_\lambda} . \tag{2.15}$$

A covariant (contravariant) tensor of the first rank (four-vector for $n = 4$) has the invariant

$$\sum_{ik} \gamma_{ik} T_i T_k$$

or

$$\sum_{ik} g_{ik}\Theta_i\Theta_k \,.$$

In ordinary relativity theory, contravariance is identical to covariance, and the above invariant becomes the square of the absolute value of the four-vector according to

$$T_x^2 + T_y^2 + T_z^2 + T_l^2 \,.$$

A covariant (contravariant) tensor of the second rank has the invariant

$$\sum_{ik} \gamma_{ik} T_{ik}$$

or,

$$\sum_{ik} g_{ik}\Theta_{ik} \,,$$

which in the case of the previous theory of relativity becomes[2]

$$T_{xx} + T_{yy} + T_{zz} + T_{ll} \,.$$

[2]In the following, we will refrain from specifying the particular form that our equations take in the case of ordinary relativity theory and will instead content ourselves with referring to the representations in [10, 14, 15, 9].

3. Differential Operations on Tensors

We introduce the following general definitions:

Definition 4 (Complementation)**.** *We designate as the extension of a covariant (contravariant) tensor of rank λ the covariant (contravariant) tensor of rank $\lambda + 1$ that is obtained from the former by "covariant (contravariant) differentiation".*

According to Christoffel (l.c.), the following holds true:

$$\left.\begin{aligned} T_{r_1 r_2 \ldots r_\lambda s} = {} & \frac{\partial T_{r_1 r_2 \ldots r_\lambda}}{\partial x_s} \\ & - \sum_k \left(\begin{Bmatrix} r_1 & s \\ k & \end{Bmatrix} T_{k r_2 r_3 \ldots r_\lambda} + \begin{Bmatrix} r_2 & s \\ k & \end{Bmatrix} T_{r_1 k r_3 \ldots r_\lambda} + \cdots \right. \\ & \left. + \begin{Bmatrix} r_\lambda & s \\ k & \end{Bmatrix} T_{r_1 r_2 \ldots k} \right) \end{aligned}\right\} \tag{3.1}$$

is a covariant tensor of rank $\lambda + 1$ which originates from the covariant tensor of rank λ. Ricci and Levi-Civita call the differential operation performed on the right-hand side of this equation the "covariant differentiation" of the tensor $T_{r_1 r_2 \ldots r_\lambda}$. The following notations has been used here:

$$\begin{Bmatrix} r & s \\ u & \end{Bmatrix} = \sum_t \gamma_{ut} \begin{bmatrix} r & s \\ t & \end{bmatrix}, \tag{3.2}$$

$$\begin{bmatrix} r & s \\ t & \end{bmatrix} = \frac{1}{2} \cdot \left(\frac{\partial g_{rt}}{\partial x_s} + \frac{\partial g_{st}}{\partial x_r} - \frac{\partial g_{rs}}{\partial x_t} \right). \tag{3.3}$$

Here,

$$\begin{bmatrix} r & s \\ t & \end{bmatrix}$$

is the Christoffel three-index symbol of the *first* kind and

$$\begin{Bmatrix} r & s \\ u & \end{Bmatrix}$$

is the Christoffel three-index symbol of the *second* kind, respectively, see chapter 8; solving Equation 3.2, one finds that[1]

$$\begin{bmatrix} r & s \\ u & \end{bmatrix} = \sum_t g_{ut} \begin{Bmatrix} r & s \\ t & \end{Bmatrix}. \tag{3.4}$$

[1] On the basis of these formulæ one can easily prove that the extension of the fundamental tensor vanishes identically.

If one replaces the covariant tensors in Equation 3.1 by the contravariant tensors reciprocal to them, one obtains as the "contravariant extension"

$$\left.\begin{aligned}\Theta_{r_1 r_2 \ldots r_\lambda s} = \sum_{ik} \gamma_{si} \Bigg(& \frac{\partial \Theta_{r_1 r_2 \ldots r_\lambda}}{\partial x_i} + \begin{Bmatrix} i & k \\ r_1 \end{Bmatrix} \cdot \Theta_{k r_2 r_3 \ldots r_\lambda} \\ & + \begin{Bmatrix} i & k \\ r_2 \end{Bmatrix} \cdot \Theta_{r_1 k r_3 \ldots r_\lambda} + \cdots \\ & + \begin{Bmatrix} i & k \\ r_\lambda \end{Bmatrix} \cdot \Theta_{r_1 r_2 \ldots k} \Bigg) \, .\end{aligned}\right\} \tag{3.5}$$

Definition 5 (Divergence)**.** *We define as the divergence of a covariant (contravariant) tensor of rank λ the covariant (contravariant) tensor of rank $\lambda - 1$ that is obtained by inner multiplication of the extension with the contravariant (covariant) fundamental tensor.*

Thus, the divergence of the covariant tensor $T_{r_1 r_2 \ldots r_\lambda}$ is the tensor

$$T_{r_1 r_2 \ldots r_\lambda} = \sum_{s r_1} \gamma_{s r_1} T_{r_1 \ldots r_\lambda s} \, , \tag{3.6}$$

and the divergence of the contravariant tensor $\Theta_{r_1 r_2 \ldots r_\lambda}$ is the tensor

$$\Theta_{r_2 r_3 \ldots r_\lambda} = \sum_{s r_1} g_{s r_1} \Theta_{r_1 \ldots r_\lambda s} \, . \tag{3.7}$$

The divergence of a tensor is not clearly apparent from this; in general, the result changes if r_1 in Equation 3.6 and Equation 3.7 is replaced by one of the indices $r_2, r_3, \ldots r_\lambda$.

Definition 6 (Generalized Laplace operation)**.** *We designate as the generalized Laplacеian operation on a tensor the sequence of extensions and the divergence. Hence, when applied to a tensor, the generalized Laplace operation produces a tensor of the same kind with the same rank.*

Of special interest are the cases where $\lambda = 0, 1, 2$.

(a) $\lambda = 0$.

The tensor we start with is a scalar T, which we can consider as a covariant or contravariant tensor of rank 0. Hence,

$$T_r = \frac{\partial T}{\partial x_r} \tag{3.8}$$

is the covariant extension of the scalar T, i.e., a covariant tensor of rank one (covariant four-vector for $n = 4$), which is called the *gradient* of the scalar. The invariant

$$\sum_{rs} \gamma_{rs} = \frac{\partial T}{\partial x_r} \frac{\partial T}{\partial x_s} \tag{3.9}$$

is the first Beltrami differential parameter of the scalar T, named after Eugenio Beltrami (16 November 1835 – 18 February 1900), see chapter 7.

To form the divergence of the gradient, one has to form from its extension

$$T_{rs} = \frac{\partial^2 T}{\partial x_r \, \partial x_s} - \sum_k \left\{ \begin{matrix} r & s \\ k \end{matrix} \right\} \frac{\partial T}{\partial x_k}$$

the scalar

$$\sum_{rs} \gamma_{rs} T_{rs} \, ,$$

which can be brought into the form (see [1], or see also the conversion of the divergence of a four-vector in the following case (b))

$$\frac{1}{\sqrt{g}} \cdot \sum_{rs} \frac{\partial}{\partial x_s} \left(\sqrt{g} \cdot \gamma_{rs} \frac{\partial T}{\partial x_r} \right) . \tag{3.10}$$

The divergence of the gradient is the result of the generalized Laplaceian operation carried out on the scalar T, and is identical with the second Beltrami differential parameter of the scalar T.

(b) $\lambda = 1$.

Let the initial tensor be a covariant four-vector, though it could just as well be a contravariant four-vector.
According to Equation 3.1, the covariant extension is

$$T_{rs} = \frac{\partial T_r}{\partial x_s} - \sum_k \left\{ \begin{matrix} r & s \\ k \end{matrix} \right\} T_k \, . \tag{3.11}$$

The divergence is

$$\sum_{rs} \gamma_{rs} T_{rs} = \sum_{rsk} \gamma_{rs} \cdot \left(\frac{\partial T_r}{\partial x_s} - \left\{ \begin{matrix} r & s \\ k \end{matrix} \right\} T_k \right) , \tag{3.12}$$

which we give the form

$$\left. \begin{aligned} \sum_{rs} \gamma_{rs} T_{rs} = \sum_{rskl} \Bigg\{ & \frac{\partial}{\partial x_s} (\gamma_{rs} T_r) - \frac{\partial \gamma_{rs}}{\partial x_s} \cdot T_r \\ & - \frac{1}{2} \cdot \gamma_{rs} \gamma_{kl} \cdot \left(\frac{\partial g_{rt}}{\partial x_s} + \frac{\partial g_{sl}}{\partial x_r} - \frac{\partial g_{rs}}{\partial x_l} \right) T_k \Bigg\} \end{aligned} \right\} \tag{3.13}$$

according to Equation 3.2.

If $\frac{\partial \gamma_{rs}}{\partial x_s}$ is eliminated using the formulæ[2]

$$\frac{\partial \gamma_{rs}}{\partial x_l} = -\sum_{\rho\sigma} \gamma_{r\rho}\gamma_{s\sigma}\frac{\partial g_{\rho\sigma}}{\partial x_l}\,, \tag{3.14}$$

the three middle terms under the sum sign in Equation 3.13 cancel each other out, leaving the expression

$$\sum_{rskl}\frac{1}{2}\cdot\gamma_{rs}\frac{\partial g_{rs}}{\partial x_l}\cdot\gamma_{kl}T_k = \sum_{kl}\gamma_{kl}T_k\cdot\frac{\partial \log(\sqrt{g})}{\partial x_l}$$

next to the first term, so that the following expression is found for the *divergence of the covariant four-vector*[3]:

$$\sum_{rs}\gamma_{rs}T_{rs} = \frac{1}{\sqrt{g}}\cdot\sum_{rs}\frac{\partial}{\partial x_s}\left(\sqrt{g}\cdot\gamma_{rs}T_r\right)\,. \tag{3.15}$$

(c) $\lambda = 2$.

Let the initial tensor be a contravariant tensor of second rank Θ_{rs}, whose complement according to Equation 3.5 is

$$\Theta_{rst} = \sum_{ik}\gamma_{ti}\left(\frac{\partial \Theta_{rs}}{\partial x_i} + \left\{\begin{matrix} i\ k \\ r \end{matrix}\right\}\cdot\Theta_{ks} + \left\{\begin{matrix} i\ k \\ s \end{matrix}\right\}\cdot\Theta_{rk}\right)\,. \tag{3.16}$$

This results in either the row divergence

$$\Theta_r = \sum_{st} g_{st}\Theta_{rst} = \sum_{sk}\left(\frac{\partial \Theta_{rs}}{\partial x_s} + \left\{\begin{matrix} s\ k \\ r \end{matrix}\right\}\cdot\Theta_{ks} + \left\{\begin{matrix} s\ k \\ s \end{matrix}\right\}\cdot\Theta_{rk}\right)\,, \tag{3.17}$$

or the column divergence

$$\Theta_s = \sum_{rt} g_{rt}\Theta_{rst} = \sum_{rk}\left(\frac{\partial \Theta_{rs}}{\partial x_r} + \left\{\begin{matrix} r\ k \\ r \end{matrix}\right\}\cdot\Theta_{ks} + \left\{\begin{matrix} r\ k \\ s \end{matrix}\right\}\cdot\Theta_{rk}\right)\,, \tag{3.18}$$

[2]This formulæ, which we also use for the derivation of the differential equations of the gravitational field in chapter 5, will be proved the following way:

Proof. Let

$$\sum_l g_{il}\gamma_{kl} = \delta_{ik} \quad (0 \text{ or } 1)\,,$$

hence

$$\sum_l g_{il}\frac{\partial \gamma_{kl}}{\partial x_t} = -\sum_l \gamma_{kl}\frac{\partial g_{il}}{\partial x_t}\,,$$

where t is some arbitrary number of $1, 2, 3, \ldots, n$.
For a definite k we thus obtain n equations in this way ($i = 1, 2, 3, \ldots, n$) with n unknowns $\frac{\partial \gamma_{kl}}{\partial x_t}$, $l = 1, 2, 3, \ldots, n$, which when solved result in the formulæ of the text. □

[3]Kottler, [8], page 21, obtains the same result starting with a special tensor of rank three (see chapter 4 in this work) with the help of the theory of integral forms.

as the divergence of the contravariant tensor Θ_{rs}, two differential operations that coincide for symmetric tensors. Because

$$\sum_r \left\{ \begin{matrix} r & k \\ r \end{matrix} \right\} = \sum_{rs} \gamma_{rs} \left[\begin{matrix} r & k \\ s \end{matrix} \right] = \sum_{rs} \frac{1}{2} \cdot \gamma_{rs} \frac{\partial g_{rs}}{\partial x_k} = \frac{\partial \log(\sqrt{g})}{\partial x_k} \tag{3.19}$$

holds true, Equation 3.18 can also be summarized in the form

$$\Theta_s = \sum_r \frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x_r} \left(\sqrt{g} \cdot \Theta_{sr} \right) + \sum_{rk} \left\{ \begin{matrix} r & k \\ s \end{matrix} \right\} \cdot \Theta_{rk} \,. \tag{3.20}$$

4. Special Tensors (Vectors)

Definition 7 (Special tensor)**.** *We call a covariant (contravariant) tensor special, if its components form a system of alternating functions of the basic variables.*

Accordingly, the components of a special tensor are subject to the following conditions:

Proposition 1.

1. *The relationship* $T_{r_1 r_2 \dots r_\lambda} = 0$ *applies if two of the indices* $r_1, r_2, \dots r_\lambda$ *are equal to each other.*
2. *If* $r_1, r_2, \dots r_\lambda$ *and* $s_1, s_2, \dots s_\lambda$ *differ only in the sequence of the indices, then* $T_{r_1 r_2 \dots r_\lambda} = \pm T_{s_1 s_2 \dots s_\lambda}$*, depending on whether* $r_1, r_2, \dots r_\lambda$ *and* $s_1, s_2, \dots s_\lambda$ *are permutations of the same class or not. As is well known, two permutations belong to the same class if both are formed from the basic permutation* $1, 2, \dots, n$ *by means of an even or an odd number of mere interchanges of two indices.*

The number of linearly independent components of a special tensor of rank λ *is thus* $\binom{n}{\lambda}$[1]

Due to these properties, the theory of special tensors is simpler but also richer than that of general tensors; it is of particular importance for mathematical physics because the theory of vectors of type λ (four-vectors, six-vectors for $n = 4$) can be traced back to that of special tensors of rank λ. From the point of view of general theory, it is more appropriate to start with the tensors themselves and treat the vectors merely as special tensors.
Important for the vector analysis of the n-dimensional manifold according to

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\, \mathrm{d}x_\mu\, \mathrm{d}x_\nu$$

is a special tensor of rank n, which is linked to the discriminant g of the line element[2]. As is well known, this discriminant transforms according to the equation

$$g' = \mathbf{J}^2 \cdot g\,, \tag{4.1}$$

[1]The coefficient $\binom{n}{\lambda}$ can be computed by the multiplicative formula

$$\binom{n}{\lambda} = \frac{n \times (n-1) \times \cdots \times (n-\lambda+1)}{\lambda \times (\lambda-1) \times \cdots \times 1},$$

which using factorial notation can be compactly expressed as

$$\binom{n}{\lambda} = \frac{n!}{\lambda!(n-\lambda)!}.$$

See `https://en.wikipedia.org/wiki/Binomial_coefficient` for binomial coefficients.
[2]The "system ϵ" of Ricci and Levi-Civita, l.c. pag. 135.

where

$$\mathbf{J} = |p_{ik}| = \left| \frac{\partial x_i}{\partial x'_k} \right|$$

is the functional determinant or JACOBI matrix (see page 76) of the substitution. If we assign a specific sign to $\sqrt{g}$ for the original reference system and define in a specific way whether this sign should change during a transformation depending on whether the substitution determinant p is negative or positive, then the equation

$$\sqrt{g'} = \mathbf{J} \cdot \sqrt{g} \tag{4.2}$$

has an exact meaning including the sign.

Let $\delta_{r_1 r_2 \ldots r_n}$ be equal to zero if two of the indices are equal to each other, and equal to ± 1 if this is not the case and the permutation $r_1, r_2, \ldots, r_n$ results from an even or odd number of permutations of two indices from the basic permutation $1, 2, \ldots, n$. Then

$$e_{r_1 r_2 \ldots r_n} = \delta_{r_1 r_2 \ldots r_n} \cdot \sqrt{g} \tag{4.3}$$

are the components of a special covariant tensor of rank n, which we will call the *covariant discriminant tensor*. This is because a transformation first yields

$$\begin{aligned} e'_{r_1 r_2 \ldots r_n} &= \delta_{r_1 r_2 \ldots r_n} \cdot \sqrt{g'} \\ &= \delta_{r_1 r_2 \ldots r_n} \cdot \mathbf{J} \cdot \sqrt{g}\,; \end{aligned}$$

and since

$$\begin{aligned} \mathbf{J} &= \sum_{i_1 i_2 \ldots i_n} \delta_{i_1 i_2 \ldots i_n} \cdot p_{i_1 1} p_{i_2 2} p_{i_3 3} \ldots p_{i_n n} \\ &= \delta_{r_1 r_2 \ldots r_n} \cdot \sum_{i_1 i_2 \ldots i_n} \delta_{i_1 i_2 \ldots i_n} \cdot p_{i_1 r_1} p_{i_2 r_2} p_{i_3 r_3} \ldots p_{i_n r_n}\,, \end{aligned}$$

it follows that

$$e'_{r_1 r_2 \ldots r_n} = \sqrt{g} \cdot \sum_{i_1 i_2 \ldots i_n} \delta_{i_1 i_2 \ldots i_n} \cdot p_{i_1 r_1} p_{i_2 r_2} p_{i_3 r_3} \ldots p_{i_n r_n}\,,$$

i.e., according to the definition in Equation 4.3, the following expression:

$$e'_{r_1 r_2 \ldots r_n} = \sum_{i_1 i_2 \ldots i_n} e_{i_1 i_2 \ldots i_n} \cdot p_{i_1 r_1} p_{i_2 r_2} p_{i_3 r_3} \ldots p_{i_n r_n}\,.$$

For the reciprocal contravariant tensor, Equation 2.13 gives the expression

$$\begin{aligned} \epsilon_{i_1 i_2 \ldots i_n} &= \sum_{r_1 r_2 \ldots r_n} \gamma_{i_1 r_1} \gamma_{i_2 r_2} \gamma_{i_3 r_3} \ldots \gamma_{i_n r_n} \cdot e_{r_1 r_2 \ldots r_n}\,, \\ \epsilon_{i_1 i_2 \ldots i_n} &= \sqrt{g} \cdot \sum_{r_1 r_2 \ldots r_n} \delta_{r_1 r_2 \ldots r_n} \cdot \gamma_{i_1 r_1} \gamma_{i_2 r_2} \gamma_{i_3 r_3} \ldots \gamma_{i_n r_n}\,, \\ \epsilon_{i_1 i_2 \ldots i_n} &= \delta_{i_1 i_2 \ldots i_n} \cdot \sqrt{g} \cdot \sum_{r_1 r_2 \ldots r_n} \delta_{r_1 r_2 \ldots r_n} \cdot \gamma_{i_1 r_1} \gamma_{i_2 r_2} \gamma_{i_3 r_3} \ldots \gamma_{i_n r_n}\,. \end{aligned}$$

However, since the determinant of the normalized subdeterminants γ_{ik} has the form

$$|\gamma_{ik}| = \frac{1}{g},$$

the expression

$$\epsilon_{i_1 i_2 \ldots i_n} = \frac{\delta_{i_1 i_2 \ldots i_n}}{\sqrt{g}} \tag{4.4}$$

follows.

The significance of the covariant (contravariant) discriminant tensor lies in the fact that its inner multiplication with a contravariant (covariant) tensor of rank λ yields a similar tensor of rank $\lambda - n$, whereby the tensor becomes of opposite type if $\lambda - n$ is negative. (Complement of the tensor.). If

$$n = 4,$$

special tensors exist up to the fourth rank, since all special tensors of higher rank vanish identically. The non-vanishing components of a special covariant tensor of the fourth rank are all equal to each other or opposite equal. The complementation, i.e., the inner multiplication with the contravariant discriminant tensor, yields a scalar, so that the differential operations that can be performed on a special tensor of the fourth rank are thus reduced to the differential operations on a scalar.

Proposition 2. *The complement of a special covariant tensor of third rank is a contravariant vector of the first kind.*

Proposition 3. *The complement of a special covariant tensor of second rank is a contravariant special tensor of second rank.*

Proposition 4. *The forming of the complement of a special covariant vector of the first kind leads to a contravariant tensor of third rank.*

The investigation of the influence of the gravitational field on physical events (see chapter 7) requires a more detailed treatment of special second-rank tensors (six-vectors).
If $\Theta_{\mu\nu}$ is a special second-rank tensor, its divergence (see Equation 3.20)

$$\Theta_\mu = \sum_\nu \frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x_\nu} \left(\sqrt{g} \cdot \Theta_{\mu\nu}\right) + \sum_{\nu\varkappa} \begin{bmatrix} \nu\ \varkappa \\ \mu \end{bmatrix} \cdot \Theta_{\nu\varkappa}$$

reduces to

$$\Theta_\mu = \sum_\nu \frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x_\nu} \left(\sqrt{g} \cdot \Theta_{\mu\nu}\right) \tag{4.5}$$

because of

$$\Theta_{\nu\varkappa} = -\Theta_{\varkappa\nu},$$
$$\Theta_{\nu\nu} = 0.$$

We further derive the dual contravariant second-rank tensor $\Theta^\star_{rs}$ from a contravariant second-rank tensor $\Theta_{\mu\nu}$ as follows.

First, we form the complement[3]

$$T_{ik} = \frac{1}{2} \cdot \sum_{\mu\nu} e_{ik\mu\nu} \cdot \Theta_{\mu\nu} \tag{4.6}$$

of the tensor, or

$$\left.\begin{aligned} T_{12} &= \sqrt{g} \cdot \Theta_{34}\,, \\ T_{13} &= \sqrt{g} \cdot \Theta_{42}\,, \\ T_{14} &= \sqrt{g} \cdot \Theta_{23}\,, \\ T_{23} &= \sqrt{g} \cdot \Theta_{14}\,, \\ T_{24} &= \sqrt{g} \cdot \Theta_{31}\,, \\ T_{34} &= \sqrt{g} \cdot \Theta_{12}\,. \end{aligned}\right\} \tag{4.6a}$$

The dual tensor we are looking for is now reciprocal to this complement and is therefore

$$\begin{aligned} \Theta^{\star}_{rs} &= \sum_{ik} \gamma_{ir}\gamma_{ks} \cdot T_{ik} \\ &= \frac{1}{2} \cdot \sum_{ik\mu\nu} \gamma_{ir}\gamma_{ks} e_{ik\mu\nu} \cdot \Theta_{\mu\nu}\,. \end{aligned} \tag{4.7}$$

The order of the two operations –complement and formation of the reciprocal tensor– can be interchanged due to the reciprocity of the two discriminant tensors.

[3]The factor $\frac{1}{2}$ serves to simplify the result without being of any importance in an invariant-theoretical way.

5. Mathematical Supplements to the Physical Part 1

5.1. Proof of the Covariance of the Momentum-Energy Equations

Theorem I. It must be proven that the Equation 5.1 of Part I, which, apart from a factor $\sqrt{-1}$, are as follows:

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu} \left(\sqrt{g} \cdot g_{\sigma\mu} \cdot \Theta_{\mu\nu}\right) - \frac{1}{2}\sqrt{g} \cdot \sum_{\mu\nu} \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \cdot \Theta_{\mu\nu} = 0\,,$$

behave covariantly with respect to any transformations, where $\sigma = 1, 2, 3, 4$.
According to Equation 3.20, the divergence of the contravariant tensor $\Theta_{\mu\nu}$ is

$$\Theta_\mu = \sum_\nu \frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x_\nu} \left(\sqrt{g} \cdot \Theta_{\mu\nu}\right) + \sum_{\nu k} \begin{Bmatrix} \nu\ k \\ \mu \end{Bmatrix} \cdot \Theta_{\nu k}\,.$$

The covariant vector T_σ reciprocal to this contravariant vector Θ_μ is thus

$$\begin{aligned} T_\sigma &= \sum_\mu g_{\sigma\mu} \cdot \Theta_\mu \\ &= \sum_{\mu\nu k} \left(\frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x_\nu} \left(\sqrt{g} \cdot g_{\sigma\mu} \cdot \Theta_{\mu\nu}\right) \right. \\ &\left. - \frac{\partial g_{\sigma\mu}}{\partial x_\nu} \cdot \Theta_{\mu\nu} + g_{\sigma\mu} \begin{Bmatrix} \nu\ k \\ \mu \end{Bmatrix} \cdot \Theta_{\nu k} \right)\,. \end{aligned}$$

But the last term of this sum is equal to

$$\sum_{\nu k} \begin{bmatrix} \nu\ k \\ \sigma \end{bmatrix} \cdot \Theta_{\nu k} = \sum_{\mu\nu} \frac{1}{2} \cdot \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \right) \cdot \Theta_{\mu\nu}\,.$$

Hence, we end up with

$$T_\sigma = \sum_{\mu\nu} \frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x_\nu} \left(\sqrt{g} \cdot g_{\sigma\mu} \cdot \Theta_{\mu\nu}\right) - \frac{1}{2} \cdot \sum_{\mu\nu} \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \cdot \Theta_{\mu\nu}\,,$$

i.e., the left side of the investigated equation, up to the factor $\frac{1}{\sqrt{g}}$. □

Thus, if that equation is divided by $\sqrt{g}$, then its left side represents the σ-component of a covariant vector, and is, therefore, in fact, covariant. For that reason, the content of those four

equations can also be expressed as a consequence of section 5.1 in the following way:

Corollary 5. *The divergence of the (contravariant) stress-energy tensor of the material stream or of the physical event vanishes.*

5.2. Differential Tensors of a Manifold Given by Its Line Element

The problem of constructing the differential equations of a gravitational field (see chapter 6) draws one's attention to the *differential invariants* and *differential covariants* of the quadratic differential form

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu\,.$$

In the sense of our general vector analysis, the theory of these differential covariants leads to the *differential tensors* that are given with a gravitational field. The complete system of these differential tensors (with respect to arbitrary transformations) goes back to a covariant differential tensor of fourth rank found by Riemann, [13] and, independently of the latter, by Christoffel, [2]. This covariant differential tensor we shall call the Riemann differential tensor or Riemann curvature tensor[1], respectively, and it is given as:

$$\left.\begin{aligned} R_{iklm} &= (ik,\ lm) \\ &= \frac{1}{2}\cdot\left(\frac{\partial^2 g_{im}}{\partial x_k\,\partial x_l} + \frac{\partial^2 g_{kl}}{\partial x_i\,\partial x_m} - \frac{\partial^2 g_{il}}{\partial x_k\,\partial x_m} - \frac{\partial^2 g_{mk}}{\partial x_l\,\partial x_i}\right) \\ &\quad + \sum_{\rho\sigma}\gamma_{\rho\sigma}\cdot\left(\begin{bmatrix} i\ m \\ \rho \end{bmatrix}\begin{bmatrix} k\ l \\ \sigma \end{bmatrix} - \begin{bmatrix} i\ l \\ \rho \end{bmatrix}\begin{bmatrix} k\ m \\ \sigma \end{bmatrix}\right). \end{aligned}\right\} \tag{5.1}$$

Through covariant algebraic and differential operations, the complete system of differential tensors (and thus also of differential invariants) of the manifold is obtained from Riemann's differential tensor and the discriminant tensor according Equation 4.3.

The $(ik,\ lm)$ are also called the Christoffel four-index symbols of the first type. In addition to Christoffel's four-index symbols of the first type, the four-index symbols of the second type are also important, namely:

$$\left.\begin{aligned} \{ik,\ lm\} &= \frac{\partial\begin{Bmatrix} i\ l \\ k \end{Bmatrix}}{\partial x_m} - \frac{\partial\begin{Bmatrix} i\ m \\ k \end{Bmatrix}}{\partial x_l} \\ &\quad + \sum_{\rho}\left(\begin{Bmatrix} i\ l \\ \rho \end{Bmatrix}\begin{Bmatrix} \rho\ m \\ k \end{Bmatrix} - \begin{Bmatrix} i\ m \\ \rho \end{Bmatrix}\begin{Bmatrix} \rho\ l \\ k \end{Bmatrix}\right), \end{aligned}\right\} \tag{5.2}$$

[1] E.g. see https://en.wikipedia.org/wiki/Riemann_curvature_tensor

which are related to the former in the following way:

$$\left.\begin{aligned} \{i\rho,\ lm\} &= \sum_k \gamma_{\rho k}\ (ik,\ lm)\ \text{, or solved} \\ (ik,\ lm) &= \sum_\rho g_{k\rho}\ \{i\rho,\ lm\}\ . \end{aligned}\right\} \tag{5.3}$$

In general vector analysis, the four-index symbols of the second type have the meaning of the components of a mixed tensor, which is covariant with respect to the third and contravariant with respect to the first rank[2].
The outstanding importance of these concepts for the differential geometry[3] of a manifold given by its line element makes it *a priori* likely that these general differential tensors are also likely to be important for the problem of differential equations of a gravitational field. In fact, it is initially possible to specify an expression for a covariant differential tensor of second rank and second order G_{im} that could enter into those equations, namely

$$G_{im} = \sum_{kl} \gamma_{kl}(ik,\ lm) = \sum_k \{ik,\ km\}\ . \tag{5.4}$$

However, it turns out that in the special case of an infinitely weak, static gravitational field, this tensor cannot be reduced to the expression $\Delta\varphi$. We must therefore leave open the question of the extent to which the general theory of differential tensors associated with a gravitational field is related to the problem of the gravitational equations. Such a connection would have to exist if the gravitational equations allowed arbitrary substitutions; but in this case alone, it seems impossible to find second-order differential equations. If, on the other hand, it were certain that the gravitational equations only allow a certain group of transformations, it would be understandable if the differential tensors provided by the general theory were not sufficient. As explained in the physics section, we are not in a position to comment on these questions.–

5.3. On the Derivation of the Gravitational Equations

The derivation of the gravitational equations described by Einstein (see chapter 6) is carried out step by step in the following way:

We start with the term

$$\mathbf{U} = \sum_{\alpha\beta\mu\nu} \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) \tag{5.5}$$

that can be expected with certainty in the energy balance and transform it by partial integration[4].

[2]This follows from the first equation of Equation 5.3

[3]The identical vanishing of the tensor R_{iklm} represents the necessary and sufficient condition for the differential form to be transformed into the form $\sum_i \mathrm{d}x_i^2$.

[4]The derivation of the required identity is simplified if we place the factor $\sqrt{g}$ under the differentiation sign, without the result being dependent on this.

This gives us

$$\mathbf{U} = \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \frac{\partial\, g_{\mu\nu}}{\partial x_\sigma} \right) - \sum_{\alpha\beta\mu\nu} \sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \cdot \frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma\, \partial x_\alpha} \,.$$

The first of the sums on the right-hand side has the desired form of a sum of differential quotients and is denoted by **A**, so that

$$\mathbf{A} = \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \right) . \tag{5.6}$$

In the second of the sums on the right-hand side, we again perform a partial integration. Then the identity reads as

$$\begin{aligned} \mathbf{U} = \mathbf{A} - \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\sigma} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \right) \\ + \sum_{\alpha\beta\mu\nu} \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \cdot \frac{\partial}{\partial x_\sigma} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) . \end{aligned}$$

The first of the sums on the right-hand side can be written as a sum of differentials and is denoted by

$$\mathbf{B} = \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\sigma} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \cdot \frac{\partial g_{\mu\nu}}{\partial x_\alpha} \right) . \tag{5.7}$$

In the second sum, we perform a differentiation. This gives us

$$\begin{aligned} \mathbf{U} = \mathbf{A} - \mathbf{B} + \sum_{\alpha\beta\mu\nu} \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \left(\gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \cdot \frac{\partial \sqrt{g}}{\partial x_\sigma} \right. \\ \left. + \sqrt{g} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \cdot \frac{\partial \gamma_{\alpha\beta}}{\partial x_\sigma} + \sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial^2 \gamma_{\mu\nu}}{\partial x_\beta\, \partial x_\alpha} \right) , \end{aligned}$$

or, if we apply Equation 3.14 to the second summand and perform partial integration in the third summand:

$$\begin{aligned}\mathbf{U} = \mathbf{A} - \mathbf{B} + \\ + \sum_{\alpha\beta\mu\nu ik} \gamma_{\alpha\beta} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \cdot \frac{\sqrt{g}}{2} \cdot \gamma_{ik} \cdot \frac{\partial g_{ik}}{\partial x_\sigma} \\ - \sum_{\alpha\beta\mu\nu ik} \sqrt{g} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \cdot \gamma_{\alpha i}\gamma_{\beta k} \cdot \frac{\partial g_{ik}}{\partial x_\sigma} \\ + \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\beta} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma} \right) \\ - \sum_{\alpha\beta\mu\nu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma} \cdot \frac{\partial}{\partial x_\beta} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \right) .\end{aligned}$$

The first two summands of the above equation have the form of terms as we set them on the left side of our identity. We now denote them by

$$\mathbf{V} = \frac{1}{2} \cdot \sum_{\alpha\beta\mu\nu ik} \frac{\partial g_{ik}}{\partial x_\sigma} \cdot \sqrt{g} \cdot \gamma_{\alpha\beta}\gamma_{ik} \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \tag{5.8}$$

and

$$\mathbf{W} = \sum_{\alpha\beta\mu\nu ik} \frac{\partial g_{ik}}{\partial x_\sigma} \cdot \sqrt{g} \cdot \gamma_{\alpha i}\gamma_{\beta k} \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} . \tag{5.9}$$

The third of the sums on the right-hand side has the form of a sum of differential quotients; if we eliminate $\frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma}$ from this sum using Equation 3.14, it turns out to be the quantity **A** already introduced with Equation 5.6. Finally, in the last sum, we also replace $\frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma}$ using the same formulæ. This gives us the expressions

$$\begin{aligned}\mathbf{U} - \mathbf{V} + \mathbf{W} = 2 \cdot \mathbf{A} - \mathbf{B} \\ + \sum_{\alpha\beta\mu\nu ik} \gamma_{\mu i}\gamma_{\nu k} \cdot \frac{\partial g_{ik}}{\partial x_\sigma} \cdot \frac{\partial}{\partial x_\beta} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \right)\end{aligned}$$

or

$$\begin{aligned}\mathbf{U} - \mathbf{V} + \mathbf{W} = 2 \cdot \mathbf{A} - \mathbf{B} \\ + \sum_{\alpha\beta\mu\nu ik} \frac{\partial g_{ik}}{\partial x_\sigma} \cdot \frac{\partial}{\partial x_\beta} \left(\sqrt{g} \cdot \gamma_{\alpha\beta}\gamma_{\mu i}\gamma_{\nu k} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \right) \\ - \sum_{\alpha\beta\mu\nu ik} \frac{\partial g_{ik}}{\partial x_\sigma} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \cdot \sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial}{\partial x_\beta} \left(\gamma_{\mu i}\gamma_{\nu k} \right) .\end{aligned}$$

The first of these sums becomes

$$- \sum_{\alpha\beta ik} \frac{\partial g_{ik}}{\partial x_\sigma} \frac{\partial}{\partial x_\beta} \left(\sqrt{g} \cdot \gamma_{\alpha\beta} \cdot \frac{\partial \gamma_{ik}}{\partial x_\alpha} \right) = -\mathbf{U}$$

due to Equation 3.14, i.e., because of

$$\sum_{\mu\nu} \gamma_{i\mu}\gamma_{\nu k} \cdot \frac{\partial g_{\mu\nu}}{\partial x_\alpha} = -\frac{\partial \gamma_{ik}}{\partial x_\alpha} \,.$$

Due to the interchangeability of the indices i and k, μ and ν, we can write the second sum as

$$\begin{aligned} 2 \cdot \mathbf{X} = 2 \cdot & \sum_{\alpha\beta\mu\nu ik} \frac{\partial g_{ik}}{\partial x_\sigma} \cdot \sqrt{g} \cdot \gamma_{\alpha\beta}\gamma_{\mu i} \cdot \frac{\partial\, g_{\mu\nu}}{\partial x_\alpha} \cdot \frac{\partial \gamma_{\nu k}}{\partial x_\beta} \\ - 2 \cdot & \sum_{\alpha\beta\mu\nu ik} \frac{\partial g_{ik}}{\partial x_\sigma} \cdot \sqrt{g} \cdot \gamma_{\alpha\beta}\, g_{\mu\nu} \cdot \frac{\partial \gamma_{i\mu}}{\partial x_\alpha} \cdot \frac{\partial \gamma_{k\nu}}{\partial x_\beta} \,. \end{aligned}$$

The identity we are looking for is therefore

$$2 \cdot \mathbf{U} - \mathbf{V} + \mathbf{W} + 2 \cdot \mathbf{X} = 2 \cdot \mathbf{A} - \mathbf{B} \,,$$

which is identical to the one given in chapter 6.

Part III.

Appendix

Most of all these input was taken from

1. https://mathworld.wolfram.com/
2. https://www.sciencedirect.com/topics/mathematics/mixed-tensor
3. https://www.tu-braunschwei*g*.de/index.php?eID=dumpFile&t=f&f=66484&token=a713aa2fb4391db085419fa632a80ec98e281a73
4. https://www.math.tu*g*raz.at/~*g*anster/lv_vektoranalysis_ss_10/20_ko-_und_kontravariante_darstellun*g*.pdf
5. https://profoundphysics.com/metric-tensor-a-complete-*g*uide-with-examples/
6. https://heiup.uni-heidelber*g*.de/catalo*g*/view/534/820/86457
7. https://www.physik.uni-hambur*g*.de/th2/lehre/dokumente/notes/art-skript-1920.pdf
8. https://www.uni-kassel.de/fb10/index.php?eID=dumpFile&t=f&f=1022&token=2883eb3002cffb0135c33c174511917d921d9b04
9. and also WIKIPEDIA

simply to gather the information, no more no less.

1. General Theory of Relativity in Short

Taken from https://www.math.ucdavis.edu/~hunter/notes/nonlinev.pdf

One of the most fundamental nonlinear field theories is the theory of general relativity (GTR). According to general relativity, space-time is a 4-dimensional (one time and three space dimensions) LORENTZian manifold, $\mathbf{M}, g_{\mu\nu}$. The curvature of space-time $\mathbf{M}$ is described by a fourth order tensor called the RIEMANN tensor, $\mathbf{Riem}(g_{\mu\nu})$. The RICCI tensor $\mathbf{Ric}(g_{\mu\nu})$ is a second order tensor obtained by contraction of the RIEMANN tensor. The scalar curvature $\mathsf{R}(g_{\mu\nu})$ is obtained by contraction from the RICCI tensor.

The EINSTEIN tensor $\mathbf{G}(g_{\mu\nu})$ is defined by

$$\mathbf{G} = \mathbf{Ric} - \frac{1}{2} \cdot \mathsf{R}(g_{\mu\nu}) \, .$$

The EINSTEIN field equations, in suitable units, are then

$$\mathbf{G} = 8\pi \cdot \mathbf{T} \, ,$$

where the energy-momentum tensor $\mathbf{T}$ describes the distribution of matter and radiation fields in space-time.

2. The Four Fundamental Interactions

In modern notation forces are viewed as *Exchange Interactions*. Exchange interactions are carried or caused by, respectively, particles with integer spin. There are four fundamental interactions known to exist: electromagnetism, weak interaction, strong interaction and gravity.

1. electromagnetic force: phonons, with spin = 0,
2. weak force or interaction: gauge boson, with spin = 1,
3. strong force or interaction: gluon, with spin = 1,
4. gravitation: graviton, with spin = 2 (a hypothesized, expspected to be massless spin-2-boson)

Note that gravity is described by a second-rank (or spin-2) tensor field.

3. Some Notes on Tensors

Note that, mathematically speaking, the mathematical object referred to in physics as a tensor is more accurately described as a tensor field.

3.1. Covariant Tensor

If T is a covariant Tensor, T is defined according to the following:

A *covariant tensor*, denoted with a lowered index (e.g., a_μ) is a tensor having specific transformation properties. In general, these transformation properties differ from those of a contravariant tensor.

To examine the transformation properties of a covariant tensor, first consider the gradient

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\hat{\mathbf{x}}_1 + \frac{\partial\phi}{\partial x_2}\hat{\mathbf{x}}_2 + \frac{\partial\phi}{\partial x_3}\hat{\mathbf{x}}_3\,, \tag{3.1}$$

for which

$$\frac{\partial\phi'}{\partial x_i'} = \frac{\partial\phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'}\,, \tag{3.2}$$

where

$$\phi\left(x_1, x_2, x_3\right) = \phi'\left(x_1', x_2', x_3'\right)\,.$$

Now let a quantity T_i be defined according to

$$T_i \equiv \frac{\partial\phi}{\partial x_i}\,, \tag{3.3}$$

then any set of quantities T_j, which transform according to the transformation law

$$T_i' = \frac{\partial x_j}{\partial x_i'}\cdot T_j \tag{3.4}$$

or, defining the $j \times i$ transformation matrix t_{ij}

$$t_{ij} \equiv \frac{\partial x_j}{\partial x_i'}\,, \tag{3.5}$$

according to

$$T_i' = t_{ij}\cdot T_j \tag{3.6}$$

is a covariant tensor. Covariant tensors are indicated with lowered indices, i.e., t_ν. To turn a

covariant tensor t_ν into a contravariant tensor t^μ (index raising), use the metric tensor $g_{\mu\nu}$ to write

$$g^{\mu\nu} t_\nu = t^\mu \,. \tag{3.7}$$

Covariant tensors are a type of tensor with differing transformation properties, denoted t_ν. However, in 3-dimensional Euclidean space,

$$\frac{\partial x_j}{\partial x_i'} = \frac{\partial x_i'}{\partial x_j} \equiv t_{ij} \tag{3.8}$$

for $i, j = 1, 2, 3$, meaning that contravariant and covariant tensors are equivalent. Such tensors are known as *Cartesian tensor*. The two types of tensors do differ in higher dimensions, however.

3.2. Contravariant Tensor

If Θ is a contravariant Tensor, Θ is defined according to the following:

A *contravariant* tensor is a tensor having specific transformation properties (cf., a covariant tensor). To examine the transformation properties of a contravariant tensor, first consider a tensor of rank 1 (a vector):

$$\mathrm{d}\mathbf{r} = \mathrm{d}x_1 \hat{\mathbf{x}}_\mathbf{1} + \mathrm{d}x_2 \hat{\mathbf{x}}_\mathbf{2} + \mathrm{d}x_3 \hat{\mathbf{x}}_\mathbf{3} \,, \tag{3.9}$$

for which

$$\mathrm{d}x_i' = \frac{\partial x_i'}{\partial x_j} \mathrm{d}x_j \,. \tag{3.10}$$

Now let a quantity Θ_i be defined according to

$$\Theta_i \equiv \mathrm{d}x_i \,,$$

then any set of quantities Θ_j which transform according to the transformation law

$$\Theta_i' = \frac{\partial x_i'}{\partial x_j} \Theta_j \,, \tag{3.11}$$

or defining the $j \times i$ transformation matrix t^{ij}

$$t^{ij} \equiv \frac{\partial x_i'}{\partial x_j} \,, \tag{3.12}$$

according to

$$\Theta_i' = t^{ij} \cdot \Theta_j \tag{3.13}$$

is a contravariant tensor. Contravariant tensors are indicated with raised indices, i.e., t^μ.

Contravariant tensors are a type of tensor with differing transformation properties, denoted t^μ. To turn a contravariant tensor t^μ into a covariant tensor t_ν (index lowering), use the *metric tensor*

$g_{\mu\nu}$ to write

$$g_{\mu\nu}\, t^{\mu} = t_{\nu}\,. \tag{3.14}$$

Covariant and contravariant indices can be used simultaneously in a mixed tensor.

Contravariant four-vectors satisfy

$$t^{\mu} = \Lambda^{\mu}_{\nu} t^{\nu}\,, \tag{3.15}$$

where Λ is a LORENTZ tensor. A LORENTZ tensor is any quantity which transforms like a tensor under the homogeneous LORENTZ transformation. In EUCLIDean spaces, and more generally in flat RIEMANNian manifolds, a coordinate system can be found where the metric tensor is constant, i.e., equal to the KRONECKER delta

$$g_{\mu\nu} = \delta_{\mu\nu}\,, \tag{3.16}$$

where

$$\delta_{\mu\nu} = \begin{cases} 0, & \text{if } \mu \neq \nu \\ 1, & \text{if } \mu = \nu\,. \end{cases}$$

Therefore, raising and lowering indices is trivial, hence covariant and contravariant tensors have the same coordinates, and can be identified. Such tensors are known as *Cartesian tensors*. The KRONECKER delta is a mixed tensor of rank 2. It is an *isotropic tensor* due to the fact that it has the same components in all viewed rotated coordinate systems.

3.3. Mixed Tensor

If $\mathfrak{I}$ is a mixed Tensor, $\mathfrak{I}$ is defined according to the following:

A *mixed* tensor is a tensor having contravariant and covariant indices. Let us consider a system, defined with respect to the systems of coordinates $\{x^k\}$ and $\{z^k\}$ by N^4 quantities $\mathfrak{I}^{ps}_{mn}$ and $\overline{\mathfrak{I}}^{kl}_{ij}$ respectively, which is transformed according to the transformation law

$$\overline{\mathfrak{I}}^{kl}_{ij} = \frac{\partial z^i}{\partial x^m}\frac{\partial z^j}{\partial x^n}\frac{\partial z^p}{\partial x^k}\frac{\partial z^s}{\partial x^l}\mathfrak{I}^{ps}_{mn}\,. \tag{3.17}$$

The system $\mathfrak{I}^{ps}_{mn}$ is called a *fourth-order mixed tensor*, that is, a second-order contravariant and second-order covariant tensor. The indices m and n are contravariant indices and indices p and s are covariant indices. In analogy with Equation 3.17 it is possible to define mixed tensors with arbitrary numbers of contravariant and covariant indices. There are two more types of fourth-order mixed tensors:

$$\mathfrak{I}^{nps}_{m}\,,\ \mathfrak{I}^{s}_{mnp}\,. \tag{3.18}$$

Using the transformation law Equation 3.17 it is easy to construct the transformation laws for arbitrary mixed tensors. The contravariant and covariant tensors can, of course, be treated as special cases of mixed tensors.
The mixed metric tensor appears as

$$g^{\nu}_{\mu} = g_{\mu\alpha}g^{\alpha\nu} = \delta^{\nu}_{\mu}\,. \tag{3.19}$$

3.4. Transformation Law

The transformation law according to

$$\overline{T}^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n} = \frac{\partial \overline{x}^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \overline{x}^{\mu_n}}{\partial x^{\alpha_n}} \frac{\partial x^{\beta_1}}{\partial \overline{x}^{\nu_1}} \cdots \frac{\partial x^{\beta_m}}{\partial \overline{x}^{\nu_m}} T^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \beta_m} \tag{3.20}$$

corresponds to the *axiomatic definition of a tensor*. The transformation according to Equation 3.20 is linear and homogen in all tensor components $T^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \beta_m}$.

4. Some Notes on Curvilinear Coordinate Systems

We assume a 4-dimensional space. It is described by the coordinate lines ξ_i with vector components i:

$$\xi_i = \left(\xi^1, \xi^2, \xi^3, \xi^4\right) . \tag{4.1}$$

Note again that the upper indices are not exponents but are indices of coordinates, coefficients or basis vectors. A special case of Equation 4.1 is the Minkowski space with coordinates x^i according to:

$$\begin{aligned} x^i &= \left(x^1, x^2, x^3, x^4\right) \\ &= (x, y, z, ct) . \end{aligned}$$

In principle, running indices are defined as follows:

$$\begin{aligned} i, j, k, \ldots &= 1, 2, 3, 4 \\ \alpha, \beta, \gamma, \ldots &= 1, 2, 3, \ldots, N \text{ where } N \in \mathbb{N} . \end{aligned}$$

In the preceeding article and hereon we make use of the important Einstein summation convention. This convention states, that over indices appearing twice in an equation, there will be summed over both indices. The summation sign is omitted, e.g.

$$\sum_{m=1}^{4} R^m_{ikm} \equiv R^m_{ikm} = R^1_{ik1} + R^2_{ik2} + R^3_{ik3} + R^4_{ik4} .$$

Let us now turn our focus to a so-called event, a definite point in space-time. An *event P* is formally a function of ξ^i, i.e.

$$P : \left(\xi^1, \xi^2, \xi^3, \xi^4\right) \text{ or shortly } \left(\xi^i\right) .$$

This is a generalization of the "point of location"; there is no such thing like a Euclidean vector, bound vector or –as it was formely known– a located vector $\vec{\xi}$.

Examples:

1. 3-dimensional space
   Relationship between curvilinear coordinates and curved space:
   a) The difference between a 2-dimensional curved space and a 2-dimensional flat space

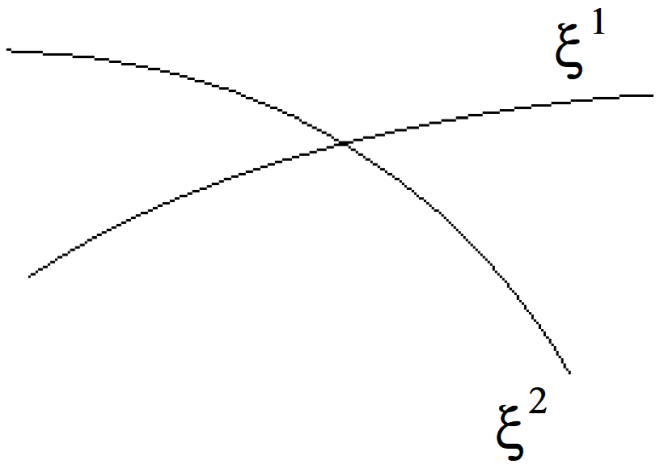

Figure 4.1.: Curvilinear coordinate lines. The ξ^1 coordinate is distinguished by the fact that all other coordinates have fixed values (ξ^2 = const, ξ^3 = const, ξ^4 = const). A different set of these constants results in a different ξ^1 coordinate. The same applies to the ξ^2 coordinate, etc.

is obvious.

b) Any 2-dimensional space that can be unfolded into a plane is flat; e.g., the surface of a cylinder or cone.

c) The surface of a sphere is curved

d) Note that curvilinear coordinates and curved space are two different things: curvilinear coordinates are also possible in flat space, e.g., plane polar coordinates.

e) Cartesian coordinates in curved space are not possible globally

f) The curvature of space is defined through the metric tensor

2. vectors in events

   a) In an event (point in 4-dimensional space), certain physical properties may be present, each of which can be described by a vector.

   b) Physical properties are, e.g. magnetic field or velocity of a particle in P or distance vector to a neighboring point P'

We now introduce basis vectors in point P in 4-dimensional space.

1. Basis vectors are different from event to event.

2. Basis vectors refer to vectors that form part of a basis P for a vector space.

3. If a base in P is defined, an arbitrary vector in P can be described with this base.

4. There exist different possibilities to introduce a base:

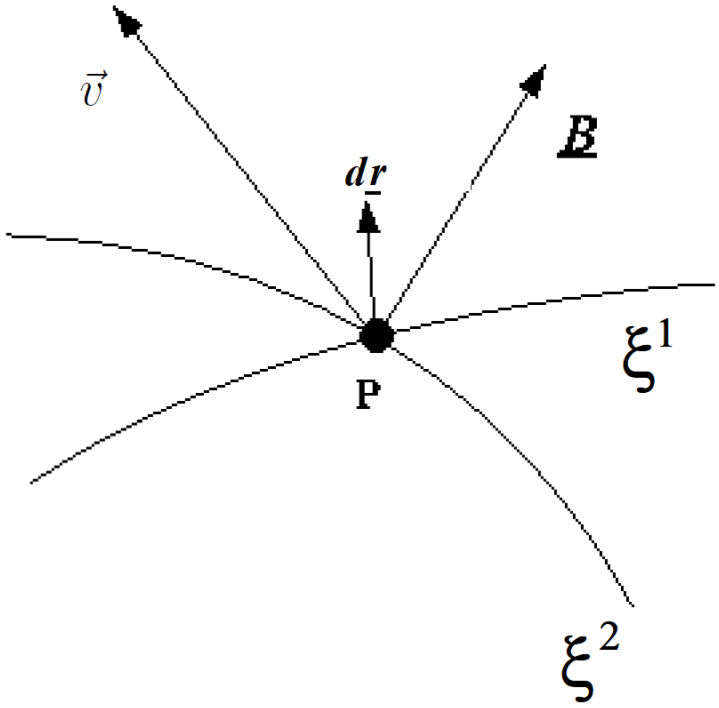

Figure 4.2.: Vectors

a) A *covariant* base $\vec{b}_i$, as is illustrated in Figure 4.3 is defined according to:

$$\vec{b}_i := \frac{\partial \vec{r}}{\partial \xi^i} \,. \tag{4.2}$$

The vectors $\vec{b}_i$ nestle against the coordinate lines or, in other words, Equation 4.2 defines the tangent vectors to the coordinate lines of ξ^i:

$$\vec{b}_1 = \left(\frac{\mathrm{d}r}{\mathrm{d}\xi^1}\right)_{\xi^2,\xi^3,\xi^4=\text{ const.}} = \frac{\partial r}{\partial \xi^1} \,.$$

Note that in the field of *covavariant* vectors (tensors of first rank) and tensors in general the *vector components* appear in the form of superscript indices.

b) A *contravariant* base $\vec{b}^j$, as is illustrated in Figure 4.4 is defined according to:

$$\vec{b}^j := \frac{\partial \xi^i}{\partial \vec{r}}$$

or, in Minkowski space

$$\vec{b}^j := \nabla \xi^i \,(x_i) \,.$$

Contravariant basis vectors play a crucial role when we deal with non-orthogonal, generalized coordinates or set of vectors, respectively. The definition of the contravariant basis vector $\vec{b}^j$ corresponds to the formation of the gradient on the surface

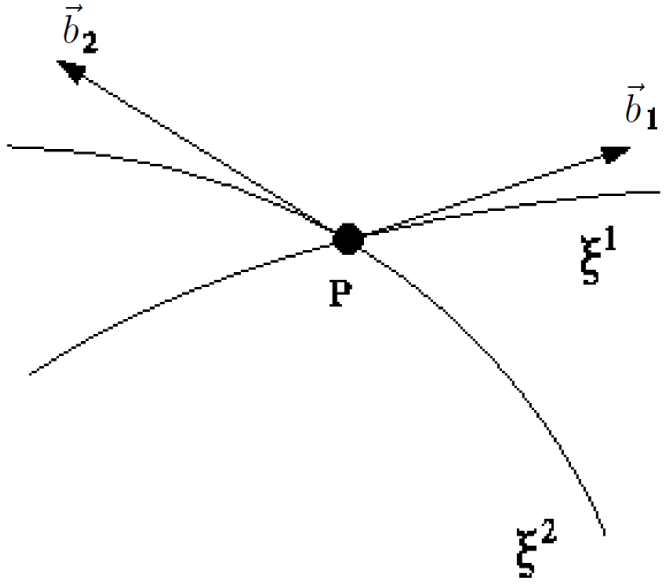

Figure 4.3.: Covariant base

$\xi^i =$ const.; then, $\vec{b}^j$ is directed orthogonal to this surface (*orthogonal vector*). Nota that in the field of *contravariant* vectors (tensors of first rank) and tensors in general the *vector components* appear in the form of subscript indices.

5. The covariant basis spans the dual space to the tangent space, i.e., the vector space spanned by the contravariant basis. For this reason, it is also said that the two bases are dual to each other (because they span dual spaces to each other). Note that in the discipline of linear algebra a dual space $V^\star$ of a vector space V over a field K is a vector space of all linear maps from V to K. If the vector space V exhibits a finite dimensionality, it has the identical dimension as his dual space $V^\star$. Both vector spaces are therefore isomorphic, i.e., they exhibit a structure-preserving mapping or morphism between themselves that can be reversed by an inverse mapping.

NOTE

A vector space is a set V of elements, called vectors, which can be added and multiplied with scalars (i.e., the numbers or elements of the field F). The field F is a set on which addition, subtraction, multiplication, and division are defined and behave as the corresponding operations on rational and real numbers.

NOTE

On a curved manifold, like that in general relativity, this vector space structure is lost because it is not clear how vectors at different points on the manifold should be added.

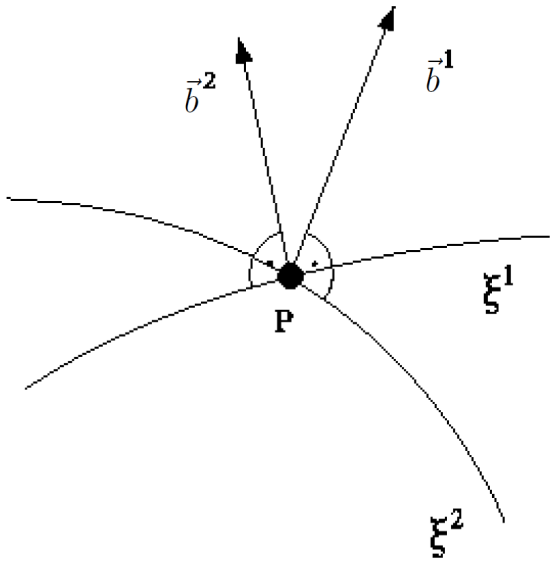

Figure 4.4.: Contravariant base

However, it still makes sense to define vectors locally in terms of infinitesimal displacements within a sufficiently small neighbourhood of a point or basis P on a manifold, which are *tangential* to the latter at P. This leads to the concept of the tangential space of a manifold, whose elements are tangential vectors, or directional derivatives of functions.

6. The basis vectors $\vec{b}_i$ and $\vec{b}^j$ are in general not normalized.

7. For orthogonal coordinates, $\vec{b}_i$ and $\vec{b}^i$ become $\vec{b}_i \parallel \vec{b}^i$.

8. The concept of tangent space plays an essential role in so-called "differentiable manifolds". An n-dimensional differentiable manifold is a space that locally (!) "looks" like an open set of $\mathbb{R}^n$ (i.e., an n-dimensional coordinate system can be introduced locally). The coordinate transformations are given by mappings that are continuously differentiable in both directions on two overlapping open sets of $\mathbb{R}^n$. In general relativity, 4-dimensional differentiable manifolds are studied (a so-called space-time continuum).

9. Orthonormality between covariant and contravariant vectors is given by

$$\begin{aligned}\vec{b}^{\,i} \cdot \vec{b}_{\,j} &= \nabla\xi^i \cdot \frac{\partial \vec{r}}{\partial \xi^j} \\ &= \left(\sum_{k=1}^{3} \frac{\partial \xi^i\,(x_1, x_2, x_3)}{\partial x_k} \cdot \hat{e}_k\right) \cdot \left(\sum_{\alpha} \frac{\partial x_\alpha\left(\xi^1, \xi^2, \ldots, \xi^N\right)}{\partial \xi^j} \cdot \hat{e}_{\,\alpha}\right) \\ &= \sum_{k=1}^{3} \frac{\partial \xi^i\,(x_1, x_2, x_3)}{\partial x_k} \frac{\partial x_k\left(\xi^1, \xi^2, \ldots, \xi^N\right)}{\partial \xi^j} \\ &= \frac{\partial \xi^i}{\partial \xi^j} = \delta^i_{\,j}\,,\end{aligned}$$

where we have used the chain rule. From this it follows that

$$\vec{b}_{\,j} \perp \vec{b}^{\,i} \text{ for } i \neq j$$

according to construction rule, see Figure 4.5. If the Kronecker delta δ is considered as a type $(1, 1)$ tensor, then the Kronecker tensor can be written $\delta^i_{\,j}$ as above with a covariant index j and contravariant index i:

$$\delta^i_{\,j} = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j). \end{cases}$$

This tensor represents:

a) The identity mapping (or identity matrix), considered as a linear mapping $V \to V$ or $V^* \to V^*$

b) The trace or tensor contraction, considered as a mapping $V^* \otimes V \to K$

c) The map $K \to V^* \otimes V$, representing scalar multiplication as a sum of outer products.

10. Both basis vector systems $\vec{b}^{\,i}$ and $\vec{b}_{\,i}$ serve as development bases. This requires the linear independence of the basis vectors.

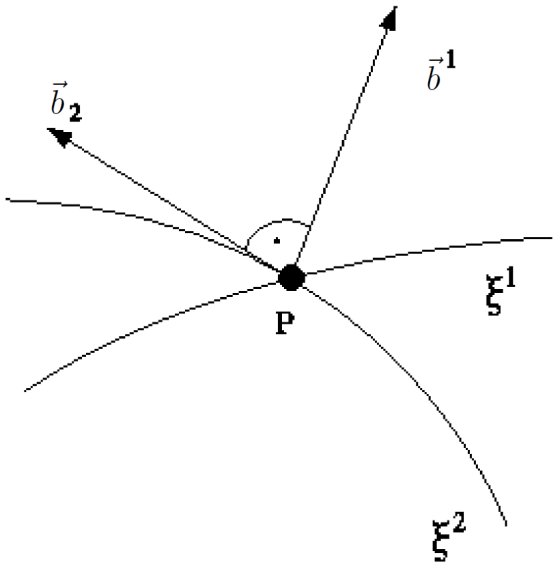

Figure 4.5.: Orthogonality of $\vec{b}_i$ and $\vec{b}^j$ at $i \neq j$ according to construction rule

5. Some Notes on the Metric Tensor

The dot product or scalar product of the basis vectors $\vec{b}$ is defined according to

$$g_{ij} \coloneqq \vec{b}_i \circ \vec{b}_j \,. \tag{5.1}$$

g_{ij} is called the fundamental *metric tensor* or simply **metric**:

$$g_{ij} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{5.2}$$

i.e., the components of the fundamental metric tensor are defined as the dot product between basis vectors.

Now, as just stated, because the metric tensor defines the dot product between basis vectors, it also defines the dot product between any other pair of vectors. This comes from the fact that we can always write two vectors in a basis P as linear combinations of the vector components v^i or u^j, respectively, and the basis vectors $\vec{b}$:

$$\begin{aligned} \vec{v} &= v^i \cdot \vec{b}_i \,, \\ \vec{u} &= u^j \cdot \vec{b}_j \,. \end{aligned}$$

The dot product between these is then:

$$\begin{aligned} \vec{v} \circ \vec{u} &= \left(v^i \circ \vec{b}_i\right) \cdot \left(u^j \circ \vec{b}_j\right) \\ &= v^i u^j \vec{b}_i \circ \vec{b}_j \,, \end{aligned}$$

which gives with the help of Equation 5.1 the final expression

$$v^i u^j g_{ij} \,. \tag{5.3}$$

Therefore, **the metric essentially determines how the dot product between any two vectors is calculated in a given coordinate system**. This is why we often say the metric defines a dot product. The metric also follows the rule

$$g_{ij} = g_{ji} \,, \tag{5.4}$$

i.e., the metric tensor is always symmetric. Further on the definitions

$$g^{ij} \coloneqq \vec{b}^i \circ \vec{b}^j \tag{5.5}$$

and

$$g_i^j := \vec{b}_i \circ \vec{b}^j \,. \tag{5.6}$$

apply.
The *inverse* metric is calculated according to:

$$\begin{aligned}\vec{b}^i &= g^{ij}\vec{b}_j \\ \vec{b}_j &= g^{jk}\vec{b}^k\end{aligned}$$

hence

$$\vec{b}^i = g^{ij} g_{jk} \vec{b}^k \stackrel{!}{=} \delta_i^k \vec{b}^k$$

and finally, the orthogonality relation

$$g^{ij} g_{jk} = \delta_i^k \tag{5.7}$$

holds. g^{ij} behaves inversely to g_{jk} when both quantities are interpreted as matrices.

With the help of the metric tensor according to Equation 5.1 we can now decompose an arbitrary vector $\vec{v}$ in a base P.

(a) According to the covariant basis

We write with the help of the summation convention

$$\vec{v} = v^i \vec{b}_i \,.$$

v^i are designated as the *contravariant components* of $\vec{v}$, see Figure 5.1.
For the contravariant components v^i it follows that

$$v^i = \vec{v}\vec{b}^i$$

because

$$\vec{v} \circ \vec{b}^i = v^j \vec{b}_j \circ \vec{b}^i$$

and with Equation 5.6 it follows that

$$\vec{v} \circ \vec{b}^i = v^j \delta_j^i = v^i \,.$$

(b) According to the contravariant basis

We write with the help of the summation convention

$$\vec{v} = v_j \vec{b}^j \,.$$

v_j are designated as the *covariant components* of $\vec{v}$, see Figure 5.2.

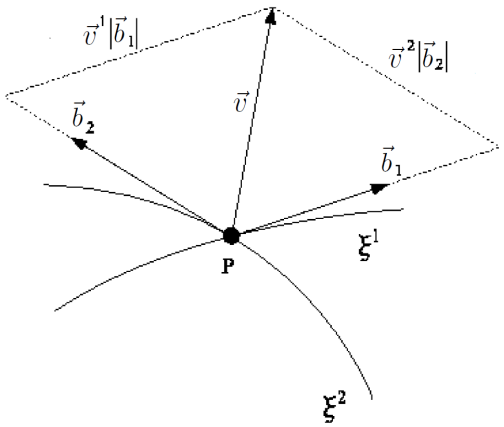

Figure 5.1.: Illustration of $\vec{v}$ in a covariant base

For the covariant components v_j it follows that

$$v_j = \vec{v}\vec{b}_j$$

because

$$\vec{v} \circ \vec{b}_j = v_k \vec{b}^k \circ \vec{b}_j$$

and with Equation 5.6 it follows that

$$\vec{v} \circ \vec{b}_j = v_k \delta^k_j = v_j \, .$$

Conversion of vector components

Covariant and contravariant components can be converted into each other. This is done with the help of the metric. We have given a vector component v^i with

$$v^i = \vec{v}\vec{b}^i = v_j \vec{b}^j \vec{b}^i \, .$$

With the help of Equation 5.5 this gives

$$v^i = g^{ij} v_j \, .$$

In other words this means:

1. Upper indices (which equals contravariant indices) can be converted to lower indices (i.e. covariant indices) using the metric, like

$$v_i = g_{ij} v^j \, .$$

2. Lower indices can be converted to upper indices using the **inverse metric** according to

$$v^i = g^{ij} v_j \, .$$

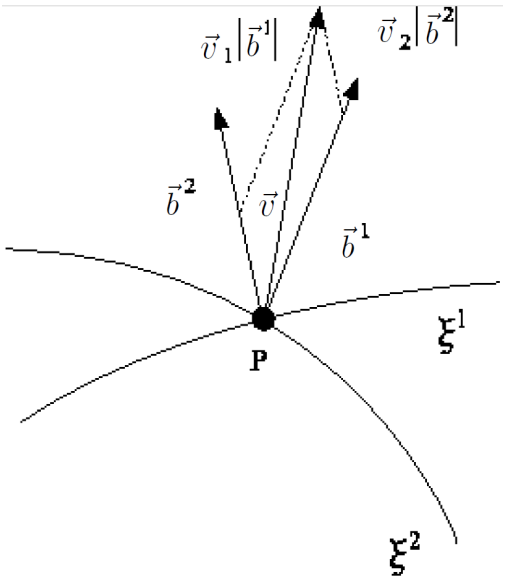

Figure 5.2.: Illustration of $\vec{v}$ in a contravariant base

Covariant and contravariant bases can be converted into each other. This is also done with the help of the metric. We have

$$\vec{v} = \left(\vec{v}\vec{b}^i\right)\vec{b}_i \,,$$

and it shall be

$$\vec{v} = \vec{b}^j \,;$$

so therefore

$$\vec{b}^j = \left(\vec{b}^j\vec{b}^i\right)\vec{b}_i = g^{ij}\vec{b}_i \,.$$

To convert a contravariant base into a Covariant one, we have

$$\vec{v} = \left(\vec{v}\vec{b}_i\right)\vec{b}^i \,,$$

and it shall be

$$\vec{v} = \vec{b}_j \,;$$

so therefore

$$\vec{b}_j = \left(\vec{b}_j\vec{b}_i\right)\vec{b}^i = g_{ij}\vec{b}^i \,.$$

6. Some Notes on Tensors in Riemannian Manifold

What is a tensor at all?

1. Tensors are objective physical quantities in an event P of a Riemannian space.
2. More abstract, tensors are physical or geometrical quantities, which components behave within a coordinate transformation from coordinate system I to II according to specific rules. The transformation takes place with the help of the Jacobi matrix $\mathbf{J}$ (see chapter 9) or the inverse of the latter for backwards transformation. The transformation matrix $\mathbf{J}$ itself is no tensor component. Thus, transformation behavior is an essential property of tensors.

We consider a 4-dimensional space with the general fundamental metric

$$\begin{aligned} \mathrm{d}s^2 &= g_{ik}\left(\xi^1, \xi^2, \xi^3, \xi^4\right) \mathrm{d}\xi^i \, \mathrm{d}\xi^k \\ &= g_{ik}(\xi) \, \mathrm{d}\xi^i \, \mathrm{d}\xi^k \, . \end{aligned}$$

With $g_{ik}(\xi) = g_{ik}$ the measurement of length and arc is determined in every event P. The following requirements must hold true:

$$\begin{aligned} g &:= \det\left(g_{ik}\right) \neq 0 \, , \\ g_{ik}(\xi) &\text{ is differentiable, which implies a differentiable manifold} \, . \end{aligned}$$

The space defined by the fundamental metric is called Riemannian space or Riemannian manifold. Riemannian space can be considered in N dimensions, whereas in general theory of relativity the case $N = 4$ is the most interesting one.

7. Some Notes on Beltrami's Differential Parameters

Let $x_1, x_2, \ldots, x_n$ be n independent variables and $\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n$ their differentials; we consider the *quadratic differential form*

$$f = \sum_{rs} a_{rs}\, \mathrm{d}x_r\, \mathrm{d}x_s \qquad (a_{rs} = a_{sr}) \;, \tag{7.1}$$

where the coefficients a_{rs} are given functions of the variables x. We assume that these functions are finite and continuous in the range of change relevant for the variables x, as are all their first and second partial differential quotients with respect to the variables x; furthermore, for this changeable range, the discriminant a of the quadratic differential form f is *always assumed to be nonzero*.

If U is an arbitrary function of $x_1, x_2, \ldots, x_n$, then we have the formulæ

$$(\mathrm{d}U)^2 = \sum_{rs} \frac{\partial U}{\partial x_r} \frac{\partial U}{\partial x_s} \tag{7.2}$$

in the square of its first differential, which is obviously a quadratic differential covariant of the given form f. If k denotes an arbitrary constant, then the differential parameter φ

$$\varphi = \sum_{rs} \left(a_{rs} + k \cdot \frac{\partial U}{\partial x_r} \frac{\partial U}{\partial x_s} \right) \mathrm{d}x_r\, \mathrm{d}x_s \tag{7.3}$$

will also be a differential covariant of f. The coefficients of the various powers of k in the quotient of the discriminant of φ and that of f will therefore all be differential parameters with the arbitrary function U. In particular, the first differential parameter, the coefficient of k itself, which we denote by $\Delta_1 U$, obviously has the value

$$\Delta_1 U = \sum_{rs} A_{rs} \frac{\partial U}{\partial x_r} \frac{\partial U}{\partial x_s} \;; \tag{7.4}$$

According to Beltrami, it is called the *first differential parameter of the function U*.

Let V now be a second arbitrary function; starting from the product of the two differentials according to

$$\mathrm{d}U\, \mathrm{d}V = \sum_{rs} \frac{\partial U}{\partial x_r} \frac{\partial U}{\partial x_s}\, \mathrm{d}x_r\, \mathrm{d}x_s \tag{7.5}$$

we again obtain a differential covariant of f, and if we replace the earlier form φ with

$$\varphi = \sum_{rs} \left(a_{rs} + k \cdot \frac{\partial U}{\partial x_r} \frac{\partial U}{\partial x_s} \right) \mathrm{d}x_r \, \mathrm{d}x_s$$

and repeat the above consideration, we see that the expression

$$\sum_{rs} A_{rs} \frac{\partial U}{\partial x_r} \frac{\partial V}{\partial x_s} \tag{7.6}$$

is a differential parameter with two arbitrary functions U and V. This differential parameter is denoted by Beltrami with the symbol

$$\nabla(U, V) \tag{7.7}$$

and is called the *mixed differential parameter of U and V*. It is obvious that if $V = U$ is set in the mixed differential parameter

$$\nabla(U, V) = \sum_{rs} A_{rs} \frac{\partial U}{\partial x_r} \frac{\partial V}{\partial x_s} \tag{7.8}$$

the first differential parameter $\Delta_1 U$ results.

8. Some Notes on the Christoffel Symbols

In the definition of the Ricci tensor according to

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} \cdot g^{\alpha\beta} R \tag{8.1}$$

or (in covariant notation)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \cdot g_{\mu\nu} R \,,$$

where $G^{\alpha\beta}$ is the Einstein manifold, $R^{\alpha\beta}$ denotes the Ricci curvature tensor, which is given by

$$\left.\begin{aligned} R_{\gamma\delta} = \left\{ \begin{matrix} \rho \\ \delta\ \rho \end{matrix} \right\}_{|\gamma} - \left\{ \begin{matrix} \rho \\ \gamma\ \delta \end{matrix} \right\}_{|\delta} + \left\{ \begin{matrix} \rho \\ \gamma\ \sigma \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ \delta\ \rho \end{matrix} \right\} \\ - \left\{ \begin{matrix} \sigma \\ \gamma\ \delta \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \sigma\ \rho \end{matrix} \right\} \end{aligned} \right\} \tag{8.2}$$

and the relation

$$R_{\gamma\delta} = g_{\alpha\gamma} g_{\delta\beta} R^{\alpha\beta} \,. \tag{8.3}$$

The symbol

$$\left\{ \begin{matrix} \alpha \\ \beta\ \gamma \end{matrix} \right\} \tag{8.4}$$

or, in modern notation

$$\left\{ \begin{matrix} \alpha \\ \beta\ \gamma \end{matrix} \right\} \equiv \Gamma^{\alpha}_{\beta\gamma} \,,$$

is a Christoffel symbol of second kind and is defined according to

$$\left\{ \begin{matrix} \alpha \\ \beta\ \gamma \end{matrix} \right\} := g^{\alpha\beta} \left[\beta\gamma, \tau\right] \,. \tag{8.5}$$

Lastly, $[\beta\gamma, \tau]$ represents a Christoffel symbol of the first kind. It is simply an abbreviated notation for the following combination of derivatives of the metric tensor, namely

$$\left[\beta\tau, \gamma\right] = \frac{1}{2} \cdot \left(g_{\beta\tau|\gamma} + g_{\gamma\tau|\beta} - g_{\beta\gamma|\tau} \right) \tag{8.6}$$

using the nomenclature

$$g_{\beta\tau|\gamma} \equiv \frac{\partial g_{\beta\tau}}{\partial x^{\gamma}} \,. \tag{8.7}$$

Written out in full, Equation 8.2 gives

$$\left.\begin{aligned} R_{\gamma\delta} = {} & g^{\rho\tau} \cdot \frac{1}{2} \cdot \left(\frac{\partial g_{\delta\tau}}{\partial x^{\rho}} + \frac{\partial g_{\rho\tau}}{\partial x^{\delta}} - \frac{\partial g_{\delta\rho}}{\partial x^{\tau}} \right) - g^{\rho\tau} \cdot \frac{1}{2} \cdot \left(\frac{\partial g_{\gamma\tau}}{\partial x^{\delta}} + \frac{\partial g_{\delta\tau}}{\partial x^{\gamma}} - \frac{\partial g_{\gamma\delta}}{\partial x^{\tau}} \right) \\ & + \left\{ g^{\rho\tau} \cdot \frac{1}{2} \cdot \left(\frac{\partial g_{\gamma\tau}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\tau}}{\partial x^{\gamma}} - \frac{\partial g_{\gamma\sigma}}{\partial x^{\tau}} \right) \right\} \\ & \cdot \left\{ g^{\sigma\tau} \cdot \frac{1}{2} \cdot \left(\frac{\partial g_{\delta\tau}}{\partial x^{\rho}} + \frac{\partial g_{\rho\tau}}{\partial x^{\delta}} - \frac{\partial g_{\delta\rho}}{\partial x^{\tau}} \right) \right\} \\ & - \left\{ g^{\sigma\tau} \cdot \frac{1}{2} \cdot \left(\frac{\partial g_{\gamma\tau}}{\partial x^{\delta}} + \frac{\partial g_{\delta\tau}}{\partial x^{\gamma}} - \frac{\partial g_{\gamma\delta}}{\partial x^{\tau}} \right) \right\} \\ & \cdot \left\{ g^{\rho\tau} \cdot \frac{1}{2} \cdot \left(\frac{\partial g_{\sigma\tau}}{\partial x^{\rho}} + \frac{\partial g_{\rho\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\rho}}{\partial x^{\tau}} \right) \right\} \end{aligned} \right\} \tag{8.8}$$

It may be noted that in general theory of relativity, where a dedicated Riemannian or pseudo-Riemannian metric on a manifold is chosen, the Riemann curvature tensor is designated as the Ricci tensor or Ricci curvature tensor. In other words, the Ricci tensor is derived from the Riemann curvature tensor for the case of a specific metric. Also, a pseudo-Riemannian manifold is called Einstein manifold, if the Ricci tensor satisfies precisely defined conditions.
Analogously to Equation 8.2 the Riemann curvature tensor ${R^{i}}_{ljk}$ is given as

$${R^{i}}_{ljk} = {\Gamma^{i}}_{jl|k} - {\Gamma^{i}}_{lk|j} + \Gamma^{i}_{mk}\Gamma^{m}_{jl} - \Gamma^{i}_{mj}\Gamma^{m}_{kl} \,. \tag{8.9}$$

9. Jacobi Determinant and Jacobi Matrix

The Jacobi matrix is a mathematical quantity that plays a role in multidimensional integral calculus, i.e., the calculation of surface and volume integrals. In vector analysis, the Jacobi matrix $\mathbf{J}$ of a vector-valued function $\mathbf{f}$ of several variables x^i is the matrix of all first-order partial derivatives of this function. The target set of the Jacobi matrix is generally a multidimensional vector space. If it is a square matrix, i.e., the function $\mathbf{f}$ has the same number of variables as the number of vector components resulting from differentiation, it is called the Jacobi determinant or Jacobi functional determinant. This is the case, for example, with the metric tensor and its matrix. In particular, the Jacobi determinant is used in the area formula and the transformation theorem derived from it.

The Jacobi matrix is denoted by $\mathbf{J}$ or also by

$$\mathbf{J}_{ik} = \frac{\partial(f_1, \ldots, f_i)}{\partial(x^1, \ldots, x^k)} = D\mathbf{f}(x)\,.$$

Let $\mathbf{f}$ be a function so that each of its first-order partial derivatives $\frac{\partial \mathbf{f}_i}{\partial x_k}$ exist on $\mathbb{R}^n$. i and k run from $1, 2, 3, \ldots n$.

For a differentiable function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$, where

$$\mathbf{f} = \mathbf{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ \vdots \\ f_i(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_k) \\ f_2(x_1, \ldots, x_k) \\ f_3(x_1, \ldots, x_k) \\ \vdots \\ f_i(x_1, \ldots, x_k) \end{pmatrix}$$

the Jacobi matrix $\mathbf{J}$ of $\mathbf{f}$ is defined as the $m \times n$ matrix whose (i, k) entry is $\frac{\partial f_i}{\partial x_k}$:

$$\mathbf{Jf}(x) = \begin{pmatrix} \operatorname{grad} f_1(x) \\ \operatorname{grad} f_2(x) \\ \operatorname{grad} f_3(x) \\ \vdots \\ \operatorname{grad} f_i(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_k}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_k}(x) \\ \frac{\partial f_3}{\partial x_1}(x) & \frac{\partial f_3}{\partial x_2}(x) & \cdots & \frac{\partial f_3}{\partial x_k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_i}{\partial x_1}(x) & \frac{\partial f_i}{\partial x_2}(x) & \cdots & \frac{\partial f_i}{\partial x_k}(x) \end{pmatrix}. \tag{9.1}$$

If in Equation 9.1 the matrix is square, that is, if the number of variables equals the number of components of function values, then its determinant is called the Jacobi determinant, which is defined as:

Definition 8 (Jacobiian)**.** *For a differentiable function* $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$*, the Jacobi determinant* $\mathbf{J}$ *is defined as the determinant of the Jacobi matrix* D *of* $\mathbf{f}$*, i.e., as*

$$\mathbf{J} \coloneqq \det\, D\mathbf{f}(x)$$

with

$$D\mathbf{f}(x) = \left(\frac{\partial f_i}{\partial x_k}(x)\right)_{i,k=1,\ldots,n} .$$

Similarly, for a second rank contravariant four-tensor A^{ik}, we can define:

Definition 9. *For a second rank contravariant four-tensor* A^{ik} *with* $A^{ik} : \mathbb{R}^n \mapsto \mathbb{R}^m$*, the functional determinant* $\mathbf{J}$ *is defined as the determinant of the Jacobi matrix* D *of* A^{ik}*, i.e., as*

$$\mathbf{J} \coloneqq \det\, DA^{ik}$$

with

$$DA^{ik} = \left(\frac{\partial A^{ik}}{\partial x^i}(x^i)\right)_{i,k=0,1,2,3} .$$

As an example, consider the function $\mathbf{f} : \mathbb{R}^2 \mapsto \mathbb{R}^2$, with $(x, y) \mapsto (f_1(x, y), f_2(x, y))$ given by

$$\mathbf{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} x^2 y \\ 5x + \sin y \end{bmatrix} .$$

This means that

$$f_1(x, y) = x^2 y$$

and

$$f_2(x, y) = 5x + \sin y$$

and the Jacobi matrix

$$\mathbf{J}_\mathbf{f}(x, y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$$

thus the Jacobi functional determinant is equal to

$$\det\left(\mathbf{J}_\mathbf{f}(x, y)\right) = 2xy \cos y - 5x^2 .$$

To express $\mathbf{J}$ by the metric tensor $g_{\mu\nu}$ or g_{ik}, respectively,we proceed as follows:
We have a completely antisymmetric pseudo unit tensor e^{iklm} and transform it into an arbitrary curvilinear coordinate system. The resulting tensor we will refer to as E^{iklm}. Further on $x\prime i$ will be Galileian and x^i will be any arbitrary curvilinear coordinates. According to the general transformation formulæ for tensors, we have

$$E^{iklm} = \frac{\partial x^i}{\partial x'^p}\frac{\partial x^k}{\partial x'^r}\frac{\partial x^l}{\partial x'^s}\frac{\partial x^m}{\partial x'^t} \cdot e^{prst} ,$$

or

$$E^{iklm} = \mathbf{J}\, e^{iklm} ,$$

where **J** is the determinant of the derivatives $\frac{\partial x^i}{\partial x'^p}$, i.e., the JACOBI functional determinant of the transformation from GALILEIan to curvilinear coordinates is

$$\mathbf{J} = \frac{\partial \left(x^0, x^1, x^2, x^3\right)}{\partial \left(x'^0, x'^1, x'^2, x'^3\right)} .$$

This functional determinant can be expressed by the determinant of the metric tensor g_{ik} (in the system of x^i, i.e., the untransformed vector functions). To do this, we write down the transformation formula for the metric tensor:

$$g_{ik} = \frac{\partial x^i}{\partial x'^l} \frac{\partial x^k}{\partial x'^m} g^{(0)lm} , \tag{9.2}$$

and set the determinants formed from the quantities on each side equal to each other. For the determinant of the tensor reciprocal to g_{ik}, the following applies according to Equation 5.7:

$$\left|g^{ik}\right| =: \frac{1}{g} . \tag{9.3}$$

The determinant of $g^{(0)lm}$ in Equation 9.2 is equal to -1. This implies that with

$$\frac{\partial x^i}{\partial x'^l} \cdot \frac{\partial x^k}{\partial x'^m} = \mathbf{J} \cdot \mathbf{J} ,$$

the relationship

$$\frac{1}{g} = -\mathbf{J}^2$$

or

$$\mathbf{J} = \frac{1}{\sqrt{-g}} . \tag{9.4}$$

applies. In curvilinear coordinates, the antisymmetric unit tensor of rank four must therefore be defined according to

$$E^{iklm} = \frac{1}{\sqrt{-g}} e^{iklm} . \tag{9.5}$$

10. And now... A Possible Final Question on the Theory of non-linear General Fractal Relativity

One may ask the question, *What is referential reality*? or how (a distorted or fractal) reality may be physically apprehended *a priori* if the $\mathrm{d}\xi^i$ are fractal coordinate differentials with a broken dimension. How and through what tool, event, etc. may the fractal differential vector operators, e.g. in a tangent space of a 4-dimensional, smooth but nowhere differentiable manifold be perceived? Is there a metric tensor consisting of some $g_{\mu\nu}$ that are continuous but nowhere differentiable functions describing a fractal, 4-dimensional space-time manifold? On a microscale, maybe perceived smooth, "harmonized" and curved *in a normal way*, but apprehended on a macroscale curved fractaly and mapped repeatedly onto itself thus giving the illusion of a non-referential (distorted) reality. The bridge between these two apperceptive worlds, macrocosm and microcosm, may be the phenomenon of scale invariance, which undergoes, possibly via a trigger or a profound change of some boundary conditions at a specific point, any form of a phase transition.

The metric may look exemplary like

$$c^2\,\mathrm{d}\tau^d = c^2\,\mathrm{d}t^2 - R^2(t)\left(\mathrm{d}r^d + S_k^2(r^d)\,\mathrm{d}\psi^2\right) .$$

The metric may also be written as

$$g_{\mu\nu} = \begin{pmatrix} \sum_{n=0}^{d} a^n \cdot \cos\left(b^n \pi x\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{r^d c^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^d & 0 \\ 0 & 0 & 0 & \left(\frac{\text{Matryoshka doll}}{S_{\text{Ma}}}\right)^{1/d_M} \end{pmatrix}$$

where d and d_M is a fractal (Matryoshka doll) dimension dependent on some kind of (mental brain lattice vibrations or environmental) state of the observer embedded in space-time. S_{Ma} is the size of the respective Matryoshka doll, M may be viewed as the brain mass and r as some kind of continuous trajectory of a Wiener process but non-continuous in time. d may in some other way be connected to Feigenbaum's constants as the evolution of a time dependent apperceptive

universe may cascade or bifurcate with the cosmic molar frequency

$$\nu_{\mathrm{cmf}} = \frac{24\hbar N \cdot \ln(\Omega)}{\pi^2 d^5 \rho} \tag{10.1}$$

in a scale invariant way towards the positive arrow of time. N is the number of particles (neurons in the brain or atoms in the universe), Ω is the number of microstates of the entity under investigation, e.g. the brain, the universe, etc. ρ is the density of the entity considered and d is a characteristic length scale in analogy to, e.g., the Reynolds number.

I discovered the group

$$\nu = \frac{N_A \cdot c^3}{\mathscr{M} \cdot G}\,, \tag{10.2}$$

where N_A is Avogadro's constant and $\mathscr{M}$ is the molar mass, during some calculations I did some time ago. Now, the group $\frac{c^3}{G}$ is part of the Bekenstein-Hawking-Entropy and with the help of the expression for the absolute value of entropy, $S = k_{\mathrm{B}} \cdot \ln(\Omega)$, the formula for the fine structure constant $\alpha = \frac{e^2}{4\pi\,\varepsilon_0\,\hbar\,c}$ and some basic chemistry formulæ the just mentioned frequency can be derived. The value ν_{cmf} for the brain is in the range of 140 mHz, if for d the average diameter of a pyramidal cell is taken, i.e. (21 µm)[1], N equals 86×10^9 neurons, Ω is twice the latter value in view that a neuron can fire or not, ρ is taken approx. as $1\,g\,\mathrm{cm}^{-3}$. If typical dimensions of the universe are inserted in Equation 10.1, ν_{cmf} is in the range of 3.8×10^{-60} Hz. Ω is chosen as a very large number, i.e. $> 1 \times 10^{300}$.
If into Equation 10.2 an average value for the molar mass $\mathscr{M}$ of neurotransmitters is inserted ($\approx 151.5\,g\,\mathrm{mol}^{-1}$), ν_{cmf} is obtained in the range of 1.60×10^{60} Hz. This value and the former value multiplied results in approximately 6, i.e. they differ with a factor of 6 or, to put it more vaguely, the two numbers are of the same order of magnitude but diametrically opposed.
Further on, with the frequency according to Equation 10.2 one can also deduce a molar fine-structure constant according to

$$\alpha_m = \frac{e^3}{\overline{q_P}^2} \cdot \frac{G}{8\pi c^2} \cdot \frac{\mathscr{M}}{2 l_P \cdot \mathscr{F}} \tag{10.3}$$

where $\mathscr{F}$ is the Faraday constant, e is the elementary charge and $\overline{q_P}$ is the reduced Planck charge, l_P is the Planck length, c the speed of light and G the gravitational constant. Another form of Equation 10.3 can be written as

$$\alpha_m = \frac{e^2}{\overline{q_P}^2} \cdot \frac{1}{2} k_{\mathrm{B}} T \cdot \frac{t_P}{8\pi\hbar}$$

where k_{B} is the Boltzmann constant and t_P the Planck time.

For $t \to \infty$ the entity is fractalo-zeroized, i.e., a final, infinitely flattened fractalized spacial quantum foam that is smooth on a microscale and in which the value for entropy is proportional to the surface of the former.

[1] see https://synapseweb.clm.utexas.edu/tutorials/morphology-dendrites/some-typical-dimensions-dendrites-few-types-neurons

It is important to realize that the Einstein field equations (or Central Fissure and Sulci Curvature-Theory equations) alone are not enough to determine the evolution of an apperceptive world underlying the gravitational and neuroendocrinic forces acting on these entities. They depend on the stress–energy (or brain-plasticity) tensor, which depends on the dynamics of matter (brain organs and neurotransmitters) and energy (such as trajectories of moving particles, molecules or growing dentritic trees), which in turn depends on the gravitational and electrophysiological field exerted (or topologically mapped onto) on the former.

One may further set Equation 10.2 equal to the frequency occurring in Eyring's transition state theory, i.e.

$$\nu = \frac{k_\mathrm{B} \cdot T}{h} \,. \tag{10.4}$$

With the definition of the Planck momentum $m_P \cdot c$, where m_P is the Planck mass, the equivalence of matter and energy $E = m_P \cdot c^2$ and the Boltzmann distribution, we finally get the nice looking formula

$$e^{-\frac{\mathrm{E}}{k_\mathrm{B} \cdot T}} = 1^{\mathbf{i}} \cdot e^{\frac{\mathscr{M}}{N_A \cdot m_P}} \,. \tag{10.5}$$

Now

$$1^{\mathbf{i}} = e^{-2k\pi} \,, k \in \mathbb{Z}$$

and the second term on the right hand side of Equation 10.5 is 1 for an average molar mass. Then we have

$$e^{-\frac{\mathrm{E}}{k_\mathrm{B} \cdot T}} \cong e^{-2k\pi} \,, \tag{10.6}$$

which suggests. . .

excellent, hehe, possibly the Boltzmann distribution may be somehow quantized in the realms of complex analysis.

11. Sketch of a 4-dimensional space-time megafractal

Figure 11.1 is a sketch of what I have drawn ten years ago or so. It was my imagination of some kind of megafractal we live in and our universe is part of. Maybe someone can imagine, what I mean with "On a microscale the arrow of time is smooth and homogeneous, flat so to say, but on a macroscale the arrow of time exhibits a fractal dimension.". The megafractal is somekind of a MENGER sponge: the surface is infinite, but in contrast to a MENGER sponge the volume tends also to infinity. The volume and surface *are* infinity itself. The distribution of black holes in this megafractal may be the result of some kind of non-linear stochastic process. Thinking about the causal relationship between the occurrence of black holes and their spatial distribution there comes to my mind a situation, when I was sitting in a garden 2023, smoking a cigarette and wondering, according to which law of nature the daisy grow and are distributed on this meadow. One may in an analogy thinking about some kind of colonization of RIEMANNian manifolds with space-time singularities. That is for sure a multi-faceted and complex process that involves, e.g., 4-dimensional space-time diffusion of particles, galaxies and stars (and also dark matter?), a thermodynamic component like some kind of supernova or radiation of surrounding stars or quasars and also gravity, which is involved in the curvature of space-time and therefore in the bending of space-time in such a manner, that curvature may be so strong that finally a space-time singularity comes into existence. Regarding the first statement on may imagine a diffusion metric (Diffusionsmetrik $\mathscr{D}_{\mu\nu}$) or a BROWNian Diffusion energy-momentum metric $\mathscr{D}_{\mathbf{T}_{\mu\nu}}$ (BROWNsche Diffusions-Energie-Impuls Metrik) in analogy to the classical metric $g_{\mu\nu}$. The $\mathscr{D}_{\mu\nu}$ or $\mathscr{D}_{\mathbf{T}_{\mu\nu}}$, respectively, are a set of cosmological diffusion differential equations or nonlinear curved evolution equations determining, e.g., the isotropy (or local anisotropy) of the universe(s) in the just mentioned megafractal.
The interstellar mass transfer coefficient β_i relating to the BROWNian Diffusion energy-momentum metric and with it the cosmological SHERWOOD number may be expressed with Equation 10.2 in the form:

$$\mathrm{Sh} = \frac{1}{\pi} \cdot \frac{\beta_i}{\mathscr{D}_{\mathbf{T}_{\mu\nu}} \cdot F_p} \cdot \frac{1}{2} k_\mathrm{B} T \,, \tag{11.1}$$

where F_p is the PLANCK force, which may be understood as the driving force for some kind of entity.

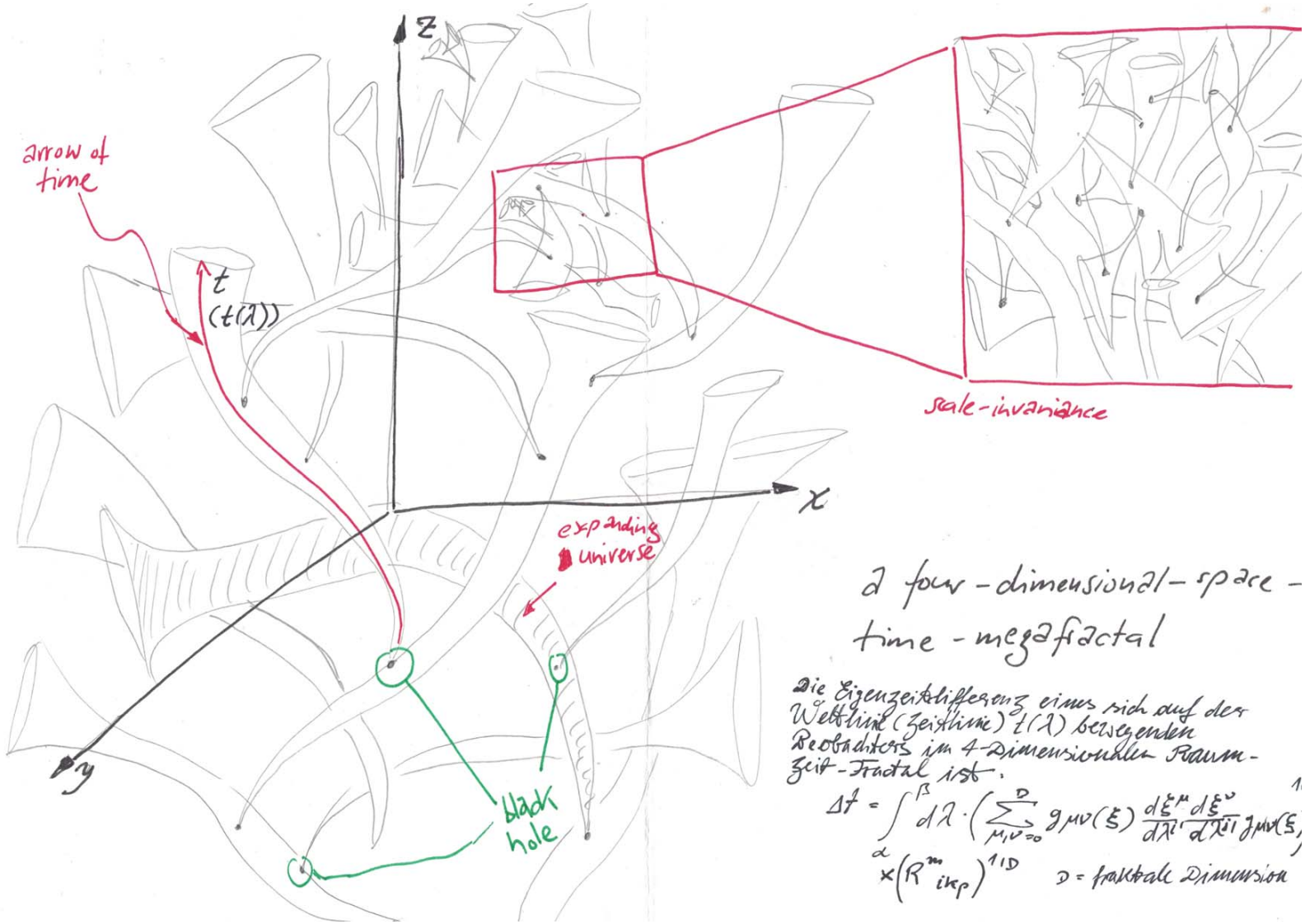

Figure 11.1.: Imaginative illustration of a 4-dimensional space-time megafractal

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Epilogue

–Finally, you get what you deserve!
You WILL be judged by HIM–

You are my most effective weapon against my most powerful and deeply insidious enemy.
Namely, YOU.